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PATH INTEGRAL: AN INVERSION OF PATH DERIVATIVES

Abstract

We introduce the concept of a path integral which integrates the path derivatives and recovers the primitives. The path integral is an extension of the Henstock integral. Moreover, we introduce the E-strong Lusin condition. Using this new concept, we give a descriptive definition of a path integral and a monotone theorem of a path differentiable function.

A.M. Bruckner, R.J. O'Malley and B.S. Thomson introduced the path derivatives which unified a number of generalized derivatives [1]. In this paper, we extend the Henstock integral to the path integral which integrates the path derivatives and recovers the primitives. The fundamental objects of our study are systems of paths on the real line.

Definition 1 *Let $x \in \mathbb{R}$. A path leading to x is a set $E_x \subset \mathbb{R}$ such that $x \in E$ and x is a point of accumulation of E_x. A system of paths is a collection $E = \{E_x : x \in \mathbb{R}\}$ such that each E_x is a path leading to x [1, p.98].*

Definition 2 *Let $F : \mathbb{R} \to \mathbb{R}$ and $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. If*

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y) - F(x)}{y - x} = f(x)$$

is finite then we say that F is E-differentiable at x and write $F'_E(x) = f(x)$ [1, p.98]. The E-continuity is similarly defined.

Definition 3 *A system of path E is said to be bilateral if every point x is a bilateral point of accumulation of E_x [1, p.100].*

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Definition 4 *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths. Then E will be said to satisfy the condition listed below if there is associated with E a positive function δ on $\mathbb{R}$ so that whenever $0 < y - x < \min(\delta(x), \delta(y))$, the set E_x and E_y intersect in the state fashion:*

4.1. Intersection condition(I.C.): $E_x \bigcap E_y \bigcap [x, y] \neq \emptyset$.

Definition 5 *A collection C of subintervals of a fixed interval $[a, b]$ is said to be an E-full cover of $[a, b]$ if there is a positive function δ on $[a, b]$ so that every interval $[y, z]$, for which $y, z \in E_x, y \leq x \leq z$ and $0 < z - y < \delta(x)$, necessarily belongs to the collection C [1, p.109].*

Theorem 1 *Let $E = \{E_x : x \in \mathbb{R}\}$ be a system of paths that is bilateral and satisfies the intersection condition. Then if C is an E-full cover of the interval $[a, b]$, C must contain a partition of every subinterval of $[a, b]$ [1, p.109].*

We first give a Riemann-type definition of a path integral.

Definition 6 *A real-valued function f is said to be E-path integrable to A on $[a, b]$ if for every $\varepsilon > 0$ there is an E-full cover C on $[a, b]$ such that for any partition $D = \{[u, v], x\}$ of $[a, b]$ from C, we have*

$$|(D) \sum f(x)(v - u) - A < \varepsilon.$$

We denote the integral A by $(EP) \int_a^b f = A$.

Remark The E-path integral includes the approximately Perron integral [4, p.138] and S-Henstock integral [3, p.156]. Clearly, the E-path integral can also recover the preponderant derivative [1, p.102, Th3.5].

Here are several obvious facts.

Fact1 Let f and g be functions mapping $[a, b]$ into $\mathbb{R}$ and let α and β be real numbers. If f and g are E-path integrable on $[a, b]$, then $\alpha f + \beta g$ is E-path integrable on $[a, b]$ and

$$(EP) \int_a^b (\alpha f + \beta g) = \alpha(EP) \int_a^b f + \beta(EP) \int_a^b g.$$

Fact2 If $f(x) = 0$ almost everywhere in $[a, b]$, then f is E-path integrabole to 0 in $[a, b]$.

Fact3 If f and g are E-path integrable on $[a, b]$ and if $f \leq g$ for almsot all x in $[a, b]$, then

$$(EP) \int_a^b f \leq (EP) \int_a^b g.$$

Fact4 (Henstock's lemma) If f is E-path intergrable on $[a,b]$ with $F(x) = (EP)\int_a^b f$, then given $\varepsilon > 0$ there is an E-full cover such that for any partition D from C, we have

$$|(D)\sum f(\xi)(v-u) - A| < \varepsilon$$

Then for any partial partition $D' = \{[u_i, v_i], \xi_i\}_{i=1}^n$ from C, we have

$$\sum_{i=1}^{n} |f(\xi)(v_i - u_i) - (F(v_i) - F(u_i))| < 2\varepsilon.$$

Theorem 2 *If f is E-path integrable on $[a,b]$, then the function $F(x) = (EP)\int_a^x f$ is E -continuous on $[a,b]$.*

PROOF. Let $x \in [a,b]$ and let $\varepsilon > 0$. Choose an E-full cover C on $[a,b]$ and let $\eta = \min\left\{\delta(x), \frac{\varepsilon}{2(1+|f(x)|)}\right\}$. Let $t \in E_x$ with $|t-x| < \eta$. Since $x \in [t,x]$ or $[x,t]$ and $[t,x],[x,t] \in C$, we can use Henstock's lemma to obtain

$$|F(t) - F(X)| \le |F(t) - F(x) - f(x)(t-x)| + |f(x)(t-x)| < \varepsilon.$$

Therefore, the function F is E-continuous at x.

Theorem 3 *If F is E-path integrable on $[a,b]$, then the function $F(x) = \int_a^x f$ is E-differentiable almost everywhere on $[a,b]$ and $F'_E(x) = f(x)$ almost every where on $[a,b]$.*

PROOF. Let X be the set of points x at which either $F_E(x)$ dose not exist or, if it does, is not equal to $f(x)$. We shall prove that X is of measure zero.

Given $\varepsilon > 0$, since f is E-path integrable on $[a,b]$. There is an E-full cover C such that for any partition $D = \{[u,v],\xi\}$ of $[a,b]$ from C, we have

$$|(D)\sum f(\xi)(v-u) - (EP)\int_a^b f| < \varepsilon. \tag{1}$$

By the Henstock's lemma, we obtain

$$(D')\sum |f(\xi)(v-u) - (F(v) - F(u)| < 2\varepsilon. \tag{2}$$

where $D' = \{[u,v],\xi\}$ is a partial partition from C.

Let $\delta(x)$ be the positive function in the definition of the E-full cover C. From the definition of X we see that every $x \in X$ there is a $\eta(x) > 0$ such that for $\delta(x) > 0$ either there is a point u with $0 < x - u < \delta(x)$, $u \in E_x$ and

$$|F(x) - F(u) - f(x)(x-u)| > \eta(x)|x-u| \tag{3}$$

or there is a point v with $o < v - x < \delta(x)$, $v \in E_x$ and

$$|F(v) - F(x) - f(x)(v - x)| > \eta(x)|v - x|. \tag{4}$$

Fix n and let X_n denote the subset of X for which $\eta(x) \geq \frac{1}{n}$. Then the above family of chosen intervals $[u, x]$ and $[x, v]$ covers X_n in the Vitali sense. Applying the Vitali covering theorem, for $\varepsilon > 0$ we can find $[u_k, v_k]$ for $k = 1, 2, \cdots, m$ with $u_k = x_k$ or $v_k = x_k$ such that

$$m^* X_n \leq \sum_{k=1}^{m} |v_k - u_k| + \varepsilon.$$

Using (3), (4), we have

$$m^* X_n \leq n \sum_{k=1}^{m} |F(v_k) - F(u_k) - f(x_k)(v_k - u_k) + \varepsilon.$$

Note that $[u_k, v_k] \in C$, for $k = 1, 2, \cdots, m$, and (2), we get

$$m^* X_n \leq (2n + 1)\varepsilon.$$

Since ε is arbitrary, the outer measure of X_n is 0 and so is X.

Definition 7 *A function F is said to satisfy the E-Strong Lusin condition, ESL, if for every set $H \subset [a, b]$ of measure zero and for every $\varepsilon > 0$ there exists an E-full cover C such that for any partial partition $D = \{[u, v], \xi\}$ from C, we have*

$$(D) \sum |F(v) - F(u)| < \varepsilon$$

where $\xi \in H$.

Remark E-strong Lusin condition originally comes from Lusin's (N) condition [7, p.224], Lee's strong Lusin condition [5, p.757] and Liao-Chew's approximately strong Lusin condition[6].

Now we prove the main result.

Theorem 4 *A function f is E-path integrable on $[a, b]$ if and only if ther exists an ESL function on $[a, b]$ such that $F_k' = f$ almost everywhere on $[a, b]$.*

PROOF. Suppose first that f is E-path integrable on $[a, b]$ and let $F(x) = (EP) \int_a^x f$. Then F is E-continuous on $[a, b]$ by Theorem 2 and $F_E' = f$ almost everywhere by Theorem 3. Let H be a set of measure zero and let

$\varepsilon > 0$. Then there is an E-full cover C shch that for any partition D of $[a,b]$ from C. We have

$$|(D)\sum f(\xi)(v-u)-(EP)\int_a^b f| < \frac{\varepsilon}{2}.$$

By the fact 2, we may suppose that $f(x) = 0$ whenever $x \in H$. By the Henstock's lemma,we have

$$\begin{aligned}&(D)\sum|F(v)-F(u)| \le (D)\sum|F(v)-F(u)-f(\xi)(v-u)| + \\ &(D)\sum|f(\xi)(v-u) \le \varepsilon\end{aligned}$$

where D is a partial partition from C.

Now suppose that there exists an ESL function F on $[a,b]$ such that $F'_E = f$ almost everywhere on $[a,b]$. Let $H = \{\xi \in [a,b]; F'_E \neq f(\xi)\}$ and let $\varepsilon > 0$. For each $\xi \in [a,b]\backslash H$ choose δ_1 so that $|f(\xi)(v-u)-(F(v)-F(u))| < \varepsilon|v-u|$ whenever $u,v \in E_\xi$ and $|u-v| < \delta_1(\xi)$. Since F satisfies the ESL conditoin, there is an E-full cover C' such that for any partial partition $D = \{[u,v],\xi\}$ from C', we have

$$(D)\sum|F(v)-F(u)| < \varepsilon$$

where $\xi \in H$. Let δ_2 be the positive function in the definition of E-full cover C'. Define

$$\delta(x) = \begin{cases} min(\delta_1(x),\delta_2(x)), & x \in [a,b]\backslash H, \\ \delta_2(x), & x \in H.\end{cases}$$

Let C be the E-full cover which is defined by the positive function δ. Suppose that $D = \{[u,v],\xi\}$ is the partition of $[a,b]$ from C, we have

$$\begin{aligned}&|\sum f(\xi)(v-u)-(F(b)-F(a))| = \\ &\quad |\sum f(\xi)(v-u)-F(v)-F(u))| \le \\ &\quad \sum_{\xi\in[a,b]\backslash H}|f(\xi)(v-u)-(F(v)-F(u))| + \\ &\quad \sum_{\xi\in H}|f(\xi)(v-u)| + \sum_{\xi\in H}|F(v)-F(u)| < \\ &\quad [1+(b-a)]\varepsilon\end{aligned}$$

where we suppose that $f(\xi) = 0$ whenever $\xi \in H$.

Following Theorem 5 is a corollary of Theorem 4 and Fact 3, but a direct proof is possible using Theorem 1.

Theorem 5 *Let F be an ESL function. If $F'_E \geq 0$ almost everywhere on $[a, b]$, then F is nondecreasing on $[a, b]$.*

PROOF. Let $H = \{\xi \in [a, b] : F'_E(\xi) \geq 0\}$ and let $\varepsilon > 0$. For every $\xi \in [a, b] \backslash H$, choose $\delta_1(\xi)$ such that

$$|F'_E(\xi)(v - u) - (F(v) - F(u))| < \varepsilon|v - u|$$

whenever $u, v \in E_\xi$ and $|v - u| < \delta_1(\xi)$. Since F satiafies the ESL condition, there is an E-full cover C' such that for any partial partition $D = \{[u, v], \xi\}$ from C', we have

$$(D) \sum |F(v) - F(u)| < \varepsilon$$

where $\xi \in H$. Let $\delta_2(\xi)$ be the positive function in the definition of E-full cover C', Define

$$\delta(x) = \begin{cases} min(\delta_1(x), \delta_2(x)), & x \in [a, b] \backslash H, \\ \delta_2(x), & x \in H. \end{cases}$$

Let C be the E-full cover which is defined by the positive functoin $\delta(\xi)$. By the theorem 1, there is a partition $D = \{[u, v], \xi\}$ from C. Labclling the D by $D = \{[a_i, a_{i+1}], \xi_i\}_{i=1}^n$, we have

$$F(b) - F(a) = \sum_{i=1}^{n} (F(a_{i+1}) - F(a_i)) = \sum_1 + \sum_2$$

where $\sum_1$ $(\sum_2)$ denote the partial sum of $\sum$ for which the associated point $\xi_i \in [a, b] \backslash H (\xi_i \in H)$. Therefore, we have

$$F(b) - F(a) \geq -\varepsilon(b - a + 1).$$

Since ε is arbitrary, we have $F(b) \geq F(a)$. For any subinterval $[\alpha, \beta]$ of $[a, b]$, same argument given that

$$F(\beta) \geq F(\alpha).$$

This shows that F is monotone nondecreasing on $[a, b]$.

Remark It is interesting to compare this monotone theorem with other ones given in [1, p.123 Cor.8.5 and Cor.8.6].

Corollary 1 *Let F be an ESL function. If $F'_E(x) = 0$ almost everywhere on $[a, b]$. Then F is constant.*

Finally, we may use theorem 12 as an alternative definition of the E-path integral.

Definition 8 *A real-valued function f on $[a,b]$ is said to be path integrable if there exists an EPI function F such that $F'_E = f$ almost everywhere on $[a,b]$. The function F is then called indefinite E-path integral of f on $[a,b]$. The increment $F(I) = F(b) - F(a)$ is termed definite E-path integral over $[a,b]$ and is denoted by*

$$(EP)\int_a^b f = F(b) - F(a).$$

Remark This gives another way which shows that E-path integral includes the S-Henstock integral [2].

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