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THE STRUCTURE OF MINIMAL ATTRACTION CENTERS OF TRAJECTORIES OF CONTINUOUS MAPS OF THE INTERVAL

Abstract

The structure of minimal attraction centers of trajectories of continuous maps of the interval is investigated. It is proved that a closed subset S of the interval is the minimal attraction center of a trajectory of a continuous map of the interval into itself if and only if S is either a nowhere dense set or a union of finitely many mutually disjoint nondegenerate closed intervals.

1 Introduction

We study the dynamics of continuous maps $f : I \to I$ where I is the interval $[0, 1]$. Under iterations of f, each point x of I generates an ordered sequence $\{f^n(x)\}_{n=1}^{\infty}$ which is called the trajectory of x. (Recall that $f^0(x) = x$ and for $n = 1, 2, 3 \ldots$ the points $f^n(x)$ are determined successively by the equality $f^n(x) = f(f^{n-1}(x))$.) The most general properties of limit behavior of the trajectory of a point x are described by its ω-limit set, i.e., by the set of limit points of the sequence $\{f^n(x)\}_{n=1}^{\infty}$. The dynamics of continuous maps on ω-limit sets and the topological structure of ω-limit sets of such maps were studied by A. N. Sharkovskiĭ in the sixties (see [5]-[8]). In particular, it was established in [5] that for continuous maps of the interval any ω-limit set is either a nowhere dense set or a finite collection of mutually disjoint nondegenerate closed intervals. Later it was proved in [1] that any set of the

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above mentioned kind is the ω-limit set of a trajectory of a continuous map of the interval.

Statistical properties of the trajectory behavior of a point x are characterized by another set which is known as the minimal attraction center or statistical limit set (σ-limit set) of x. We use the notation $\sigma(x, f)$ for this set. Informally, $\sigma(x, f)$ is the smallest closed set such that the trajectory of x moves near this set almost all the time. This set was introduced in [2] [3] (see also [4]) in the study of the existence problem for invariant measures of dynamical systems. It should be noted that the main properties of the dynamics on σ-limit sets are essentially different from the properties of the dynamics on ω-limit sets. In particular concerning the topological structure of these sets, we observe that for any infinite ω-limit set the isolated points are nonperiodic, and for any σ-limit set the isolated points must be periodic. Nevertheless it turns out that the conditions, which describe the admissible structure of σ-limit sets for continuous maps of the interval, are identical to the conditions, which describe the admissible structure of ω-limit sets. Namely in the present paper we prove that a nonempty closed subset of the interval is the σ-limit set of a trajectory of a continuous map of the interval if and only if this set is either a nowhere dense set or a finite collection of mutually disjoint nondegenerate closed intervals.

2 Main Result

We say that the trajectory of a point $x \in I$ is statistically asymptotic [3] to a closed set $F \subset X$ if for any set U open in I with $F \subset U$ we have $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_U(f^i(x)) = 1$ where χ_U is the indicator of the set U, i.e., the real-valued function on X such that $\chi_U(y) = 1$ for $y \in U$ and $\chi_U(y) = 0$ for $y \notin U$. The σ-limit set $\sigma(x, f)$ is defined to be the smallest closed set to which the trajectory of x is statistically asymptotic. The set $\sigma(x, f)$ is characterized by the following two properties:

(i) the trajectory of x is statistically asymptotic to $\sigma(x, f)$,

(ii) for every $y \in \sigma(x, f)$ and for every open set U with $y \in U$, we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_U(f^i(x)) > 0.$$

The following theorem describes the admissible structure of minimal attraction centers of trajectories of continuous maps of the interval.

Theorem 1 *A nonempty closed subset of the interval is the minimal attraction center of a trajectory of a continuous map of the interval if and only if either this subset is nowhere dense or it is a finite collection of mutually disjoint nondegenerate closed intervals.*

PROOF. It is not hard to see that for continuous maps of the interval, any minimal attraction center has the structure mentioned in the theorem. Indeed, if the minimal attraction center of a trajectory is dense in some part of the interval, then it contains a subinterval because it is closed. Therefore after a finite number of iterations the trajectory hits this subinterval and hence its minimal attraction center coincides with its ω-limit set. (We use the known facts that $\sigma(x, f) \subset \omega(x, f)$ and $f(\sigma(x, f)) = \sigma(x, f)$; see [4] for details.) By the result of [5], which has been mentioned in the previous section, we conclude that in this case the minimal attraction center is a finite collection of mutually disjoint nondegenerate closed intervals. Thus the "only if" part of the theorem is proved.

Let $\sigma\mathrm{Rec}(f)$ denote the set of all σ-recurrent points of the map f, i.e., the set of points which belong to their minimal attraction centers. In order to prove the "if" part, we prove the following auxiliary theorem first.

Auxiliary Theorem *Let $f : I \to I$ be an expanding continuous map, i.e., for any open subset U of I, there is a positive integer $K < \infty$ (depending on U) for which $f^K(U) = I$. Then for any nonempty invariant subset Z of $\sigma\mathrm{Rec}(f)$, there exists a point $x \in I$, the minimal attraction center of which is $\overline{Z}$.*

We divide the proof of this theorem into some lemmas.

Lemma 1 *For each $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that for any open interval $J \subset I$ we have $f^K(J) = I$ whenever the length of J is not less than ε.*

PROOF. If for some ε such a $K(\varepsilon)$ does not exist, then it is possible to find a sequence of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$ with $|b_n - a_n| \geq \varepsilon$ and a sequence of integers $\{K_n\}_{n=1}^{\infty}$ tending to ∞ as $n \to \infty$ such that $f^{K_n}((a_n, b_n)) \neq I$ for all n. We can assume that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. It is easy to see that for $\delta = \varepsilon/4$ we have $(a + \delta, b - \delta) \neq \emptyset$ and $f^K((a + \delta, b - \delta)) \neq I$ for all K. But this contradicts hypothesis of the theorem. □

The following lemma is immediately implied by the previous one.

Lemma 2 *Let $\varepsilon > 0$ and $K(\varepsilon)$ be determined by Lemma 1. Suppose that for some subinterval J of I and for some $n > K(\varepsilon)$, we have $f^n(J) \neq I$. Then for any $i \leq n - K(\varepsilon)$ the length of the interval $f^i(J)$ is less than ε.*

For any point $x \in I$ and any $\varepsilon > 0$ let the symbol $B(x, \varepsilon)$ denote the ε-neighborhood of the point x in I, i.e., the set $\{y \in I : |y - x| < \varepsilon\}$. The following lemma is a consequence of the obvious fact that any finite segment of a trajectory does not influence its statistical behavior.

Lemma 3 *For every point $x \in \sigma\mathrm{Rec}(f)$ and every $\varepsilon > 0$ there exist $\nu > 0$ and a sequence of positive integer numbers $\{L_m\}_{m=1}^{\infty}$ such that*

$$\frac{\mathrm{card}\{i < L_m : f^i(x) \in B(x,\varepsilon)\}}{m + L_m} > \nu$$

for all $m \geq 1$ (card$A$ denotes the number of elements in the finite set A).

PROOF. The condition $x \in \sigma\mathrm{Rec}(f)$ means that $x \in \sigma(x, f)$ and hence for any $\varepsilon > 0$ we have $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{B(x,\varepsilon)}(f^i(x)) = \nu_0(\varepsilon) > 0$. Fix ε and choose any $\nu \in (0, \nu_0(\varepsilon))$. Then fix $m \geq 1$. Let $\Delta = \nu_0(\varepsilon) - \nu$ and $\delta_k = \frac{m}{m+k}$. Also we fix an arbitrary positive integer k, for which $\delta_k < \frac{\Delta}{2}$. The existence of the above mentioned limit implies the existence of an integer $L_m > k$, for which

$$\frac{\mathrm{card}\{i < L_m : f^i(x) \in B(x,\varepsilon)\}}{L_m} > \nu + \frac{\Delta}{2}.$$

Then for this L_m we have

$$\frac{\mathrm{card}\{i < L_m : f^i(x) \in B(x,\varepsilon)\}}{m + L_m} =$$

$$= \frac{m + \mathrm{card}\{i < L_m : f^i(x) \in B(x,\varepsilon)\}}{m + L_m} - \frac{m}{m + L_m} > \nu + \frac{\Delta}{2} - \delta_k > \nu$$

that completes the proof of the lemma. □

PROOF OF THE AUXILIARY THEOREM. Let us set $\delta_i = \frac{1}{i+1}$ for $i = 1, 2, \ldots$ and consider the compact set $\overline{Z}$. For this set we can find a sequence of finite δ_i-nets $Z_i = \{z_1^{(i)}, \ldots, z_{N_i}^{(i)}\}$ consisting of points from Z and such that $Z_i \subset Z_{i+1}$ for all $i \geq 1$. Now having defined the sequence $\{N_i\}_{i=1}^{\infty}$ where N_i is the number of points in the δ_i-net Z_i, any positive integer j can be uniquely represented in the form $j = n(j) + \sum_{k<i(j)} N_k$ where $n(j)$ satisfies the condition $1 \leq n(j) \leq N_{i(j)}$. Thus all points of the finite δ_i-nets Z_i form an infinite sequence $\{z_j\}_{j=1}^{\infty}$ by the condition $z_j = z_n^{(i)}$ where n and i are equal to the above described $n(j)$ and $i(j)$ respectively. By the equality $\varepsilon_j = \delta_{i(j)}$ the sequence $\{z_j\}_{j=1}^{\infty}$ can also be associated with the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive real numbers, which will define neighborhoods of points z_j. Note that for any $j \geq 1$ there are infinitely many $k > j$ with $z_k = z_j$ and that the points z_j form a subset of Z dense in $\overline{Z}$.

We use the sequence of points $\{z_j\}_{j=1}^{\infty}$ and numbers $\{\varepsilon_j\}_{j=1}^{\infty}$ in order to construct a sequence $\{F_j\}_{j=0}^{\infty}$ of nested closed subsets of I with a nonempty intersection containing a point having the required properties.

Let $F_0 = I$ and $t_0 = 0$. By applying Lemma 3 to the point z_1 and ε_1, we define the number ν and the sequence $\{L_m\}$. Then we set ν_1 is equal to this ν and $l_1 = L_0 + 2K(\varepsilon_1)$. Now let t_1 be any positive integer such that $t_1 \geq l_1$ and $\frac{t_1-2K(\varepsilon_1)}{t_1} > 1/2$. Since the map f is continuous and expanding, we can find an interval F_1^* which contains the point z_1 and for which $f^{t_1-K(\varepsilon_1)}(\overline{F_1^*}) = \overline{B(f^{t_1-K(\varepsilon_1)}(z_1), \varepsilon_1)}$. By Lemma 1 we can find a closed interval $F_1 \subset F_1^*$ such that $f^{t_1}(F_1) = \overline{B(z_2, \varepsilon_2)}$. Note that by Lemma 2, for any $t \leq t_1 - 2K(\varepsilon_1)$, the length of the interval $f^t(F_1^*)$ is less than ε_1 and hence the first $t_1 - 2K(\varepsilon_1)$ iterations of the interval F_1 belong to ε_1-neighborhoods of the corresponding iterations of the point z_1.

By applying Lemma 3 to the point z_2 and ε_2, we define a new number ν and a new sequence $\{L_m\}$, and then set ν_2 is equal to this ν and $l_2 = L_{t_1} + 2K(\varepsilon_2)$. Now let t_2 be any positive integer such that $t_2 \geq l_1$, $t_2 \geq l_2$ and $\frac{t_2-2K(\varepsilon_2)}{t_1+t_2} > 2/3$. Since the map f is continuous and expanding and also since $f^{t_1}(F_1) = \overline{B(z_2, \varepsilon_2)}$, we can find an interval F_2^* which is contained in the interval F_1 and for which the following two conditions are satisfied:

a) the point z_2 belongs to $f^{t_1}(F_2^*)$;

b) $f^{t_1+t_2-K(\varepsilon_2)}(\overline{F_2^*}) = \overline{B(f^{t_2-K(\varepsilon_2)}(z_2), \varepsilon_2)}$.

Now by Lemma 1 we can find a closed interval $F_2 \subset F_2^*$ such that $f^{t_1+t_2}(F_2) = \overline{B(z_3, \varepsilon_3)}$.

Similarly for any $j \geq 1$, we apply Lemma 3 to the point z_j and ε_j in order to define a new number ν and a new sequence $\{L_m\}$, and then set ν_j is equal to this ν and $l_j = L_t + 2K(\varepsilon_j)$ where $t = \sum_{i<j} t_i$. Now let t_j be any positive integer such that $t_j \geq l_i$ for all $i \leq j$ and $\frac{t_j-2K(\varepsilon_j)}{t_1+t_2+\cdots+t_j} > \frac{j}{j+1}$. Since the map f is continuous and expanding and also since $f^{t_1+\cdots+t_{j-1}}(F_{j-1}) = \overline{B(z_j, \varepsilon_j)}$, we can find an interval F_j^* which is contained in the interval F_{j-1} and for which the following two conditions are satisfied:

a) the point z_j belongs to $f^{t_1+\dots t_{j-1}}(F_j^*)$;

b) $f^{t_1+\cdots+t_j-K(\varepsilon_j)}(\overline{F_j^*}) = \overline{B(f^{t_j-K(\varepsilon_j)}(z_j), \varepsilon_j)}$.

Now by Lemma 1 we can find a closed interval $F_j \subset F_j^*$ such that $f^{t_1+\dots t_j}(F_j) = \overline{B(z_{j+1}, \varepsilon_{j+1})}$.

By Lemma 2 the first $t_j - 2K(\varepsilon_j)$ of the last t_j iterations of the interval F_j belong to ε_j-neighborhoods of the corresponding iterations of the point

z_j. Since the interval F_j is contained in all previously constructed intervals, due to the properties of these intervals indicated above we can conclude that trajectories of all points of the interval F_j approximate, successively for $i = 1, \dots, j$, pieces of trajectories of length $t_i - 2K(\varepsilon_i)$ of points z_i to within ε_i.

Let us prove that for the point x, which is defined by the intersection of all closed intervals F_j as $j \to \infty$, we shall have $\sigma(x, f) = \overline{Z}$. To this end for arbitrary $\varepsilon > 0$ we consider the ε-neighborhood $B(\overline{Z}, \varepsilon)$ of the set $\overline{Z}$. It is clear that for all j greater than some $j_0 \geq 1$, we have $B(z_j, \varepsilon_j) \subset B(\overline{Z}, \varepsilon)$. Then for each $j > j_0$, by the choice of t_j, for any $t \geq \sum_{i \leq j} t_i$ the relative time of being of the trajectory of the point x outside of the set $B(\overline{Z}, \varepsilon)$ does not exceed $1/(j+1)$ and hence it decreases to 0 as $t \to \infty$. Therefore $\sigma(x, f) \subseteq \overline{Z}$. Furthermore since for any j there are infinitely many $k > j$ with $z_k = z_j$ and $\varepsilon_k < \varepsilon_j$ and also since t_j are chosen to be not less than l_i for all $i \leq j$, by Lemma 3 for each point z_j we have $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B(z_j,\varepsilon)}(f^i(x)) > 0$ for any fixed $\varepsilon > 0$. Hence for any $j \geq 1$ the point z_j belongs to $\sigma(x, f)$. Since $\sigma(x, f)$ is closed and the points z_j form a dense set in $\overline{Z}$, we obtain the inclusion $\overline{Z} \subseteq \sigma(x, f)$. This proves the equality $\sigma(x, f) = \overline{Z}$ and completes the proof of the auxiliary theorem. □

Using the Auxiliary Theorem, we can complete the proof of the theorem on the structure of σ-limit sets as follows.

Let S be a closed interval $[a, b]$ and c be the midpoint of this interval. Let f be the tent map on $[a, b]$, i.e. the continuous map such that $f(a) = f(b) = a$, $f(c) = b$ and f is linear on each of the intervals $[a, c]$, $[c, b]$. It is well known that the tent map is expanding and that periodic points of f are dense in the interval. Therefore by the Auxiliary Theorem there exists a point $x \in [a, b]$, for which $\sigma(x, f) = [a, b] = S$.

If $S = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_n, b_n]$ where $n > 1$ and $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$, then for $i = 1, 2, \dots n-1$ we set $f(a_i) = a_{i+1}$, $f(b_i) = b_{i+1}$ and $f(a_n) = f(b_n) = a_1$, $f(\frac{a_n+b_n}{2}) = b_1$. Extending f by linearity, we obtain a continuous piecewise linear map $f : I \to I$, for which $f([a_i, b_i]) = [a_{i+1}, b_{i+1}]$ for $i = 1, 2, \dots, n-1$ and $f([a_n, b_n]) = [a_1, b_1]$. The restriction of the map f^n on the interval $[a_1, b_1]$ is the tent map on $[a_1, b_1]$ and we can use the Auxiliary Theorem again in order to obtain a point $x \in [a_1, b_1]$ with $\sigma(x, f^n) = [a_1, b_1]$ and hence $\sigma(x, f) = S$.

Now suppose that S is a nonempty closed nowhere dense subset of $I = [0, 1]$. We are going to construct a continuous expanding map $f : I \to I$, for which $S \subset \mathrm{Fix}(f)$ where $\mathrm{Fix}(f)$ denotes the set of fixed points of f. We use the following construction.

Let (a, b) be a nondegenerate interval and let L and R be nonnegative real numbers. We define a real continuous function g on the segment $[a, b]$ by the following two conditions:

(i) $g(a) = a$, $g(b) = b$, $g(a+(b-a)/3) = b+R$ and $g(b-(b-a)/3) = a-L$;

(ii) g is linear on each component of the set $(a,b)\backslash\{a+(b-a)/3, b-(b-a)/3\}$.

If an interval (a,b) and nonnegative real numbers L and R are given, then we say that the function g is defined by parameters (a,b), L, R. Note that we have $|g(J)| \geq |J|$ for any interval $J \subset [a,b]$ where $|J|$ denotes the length of the interval J. Furthermore, if $g(J)$ contains no endpoints of $[a,b]$, then $|g(J)| \geq \frac{3}{2}|J|$, and if J contains an endpoint of $[a,b]$, then $[a,b] \subset f^K(J)$ for some $K \leq 1 + \log_3 \frac{b-a}{|J|}$.

Let $\Gamma = \{G_i\}$, $i \geq 0$, be the family of all components of the open dense set $I\backslash S$, which are numbered with $|G_j| \geq |G_k|$ if $j < k$, and let $\Gamma_n = \{G_i\}_{i \leq n}$. Fix a sequence $n_0 = 0 \leq n_1 \leq n_2 \leq \dots$ such that for any $k \geq 1$ the finite family Γ_{n_k} of open intervals has the following property: if we divide the interval I into 2^k equal parts, then for any such part J we can find an interval G from Γ_{n_k}, for which $G \cap J \neq \emptyset$. We can find such a sequence because the set S is nowhere dense in I. Note that for each $G_i \in \Gamma$ with $i \geq 1$, there exists a unique $k = k(i)$ such that $G_i \in \Gamma_{n_k}$ and $G_i \notin \Gamma_{n_{k-1}}$. It is clear that if Γ is infinite (i.e. if S is infinite), then $k(i) \to \infty$ as $i \to \infty$.

Let $L_0 = \inf G_0$ and $R_0 = 1 - \sup G_0$. (Recall that we construct the map on the interval $I = [0,1]$.) If $i \geq 1$, then we define nonnegative real numbers L_i and R_i as follows:

a) first we find $k = k(i)$ such that $G_i \in \Gamma_{n_k}$ and $G_i \notin \Gamma_{n_{k-1}}$;

b) if there are no elements of Γ_{n_k} to the left of G_i, then $L_i = \inf G_i$; if there are elements of Γ_{n_k} to the left of G_i, then $L_i = l_i + |G_{n_k}|$ where l_i is the distance between G_i and the interval from Γ_{n_k}, which is nearest to G_i from the left;

c) if there are no elements of Γ_{n_k} to the right of G_i, then $R_i = 1 - \sup G_i$; if there are elements of Γ_{n_k} to the right of G_i, then $R_i = r_i + |G_{n_k}|$ where r_i is the distance between G_i and the interval from Γ_{n_k}, which is nearest to G_i from the right.

Now we can define the map $f : I \to I$ by the following two conditions:

a) for $x \in S$ we set $f(x) = x$;

b) on every $G_i \in \Gamma$ the map f is equal to the function g introduced above defined by the parameters G_i, L_i, R_i.

In order to prove the continuity of the map f, it is sufficient to prove that L_i and R_i tend to zero as $i \to \infty$. By the definition of L_i we have either L_i is equal to the distance from G_i to the left end of the interval I or $L_i = l_i + |G_{n_k}|$ where l_i is the distance between G_i and the interval from Γ_{n_k}, which is nearest to G_i from the left. Having determined $k = k(i)$, by the definition of Γ_{n_k} we have $L_i \leq 2^{-k}$ in the first case and $l_i \leq 2^{-k+1}$ in the second one. Hence $L_i \leq 2^{-k+1} + |G_{n_k}|$. Since $|G_{n_k}| \to 0$ as $k \to \infty$ and $k(i) \to \infty$ as $i \to \infty$, we have $L_i \to 0$ as $i \to \infty$.

Similarly we can prove that $R_i \to 0$ as $i \to \infty$ and hence the map f is continuous.

Now let us prove that the map f is expanding. Let U be an open in I subinterval of I such that $U \cap S \neq \emptyset$. Since S is nowhere dense in I, there exists $G_i \in \Gamma$ such that $U \cap G_i \neq \emptyset$ and one of the endpoints of G_i belongs to U. Let us define $k = k(i)$ and consider the interval $J = U \cap G_i$. It is not hard to see that for some $t \leq 1 + \log_3 \frac{|I|}{|J|} + n_k(1 + \log_3 \frac{|I|}{|G_{n_k}|})$, the interval $f^t(J)$ contains an endpoint of G_0 and $|G_0 \cap f^t(J)| \geq |G_{n_k}|$. Hence for $\tau = 1 + \log_3 \frac{|I|}{|G_{n_k}|}$, we must have $f^{t+\tau}(J) = I$.

If $U \cap S = \emptyset$, then $U \subset G_i$ for some $G_i \in \Gamma$. Using the properties of the map g mentioned above and the construction of the map f, we can conclude that if for this U we have $f(U) \cap S = \emptyset$, then $|f(U)| \geq \frac{3}{2}|U|$. Therefore for a finite N we must have $f^N(U) \cap S \neq \emptyset$ and hence we can apply the arguments of the previous case to this new interval $U^* = f^N(U)$. This proves that the map f is expanding. Now applying the Auxiliary Theorem to the map f and the closed set $S \subset \mathrm{Fix}(f) \subset \sigma\mathrm{Rec}(f)$, we complete the proof.

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