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A CONVERGENCE THEOREM FOR GENERALIZED RIEMANN INTEGRALS

Abstract

A necessary and sufficient condition is given that the limit f of a sequence $\{f_n\}$ of generalized Riemann integrable functions be integrable and that $\int_I f = \lim \int_I f_n$.

In this note we will establish an elementary necessary and sufficient condition that the limit f of a sequence (f_k) of generalized Riemann (= Henstock-Kurzweil) integrable functions be integrable and that $\int_I f = \lim_k \int_I f_k$. Some examples are considered, and we will show how this theorem can be applied to prove Hake's theorem for the integral over an infinite interval $[a, \infty]$.

We assume that the reader is familiar with the elementary properties of the generalized Riemann integral; see, for example, the books [1], [4] and [5]. If $P := \{(I_i, t_i)\}_{i=1}^n$ is a partition of a closed interval $I := [a, b]$ in the extended real numbers $\overline{\mathbb{R}}$ given by the partition points $-\infty \le a = x_0 < x_1 < \cdots < x_n = b \le \infty$ and tags $t_i \in I_i := [x_{i-1}, x_i]$, then the Riemann sum of a function f corresponding to P is

$$S(f; P) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

where, as usual, we understand that $0 \cdot \infty = 0$ and that $f(\pm\infty) = 0$.

The ideas presented here are close to those in the article of R. A. Gordon [2]; in particular, the notion of γ-convergence of a sequence of functions is similar to (but considerably weaker than) Gordon's notion of a δ-Cauchy sequence. It is worth noting that the condition that we give applies to an infinite interval; in fact, it also applies to integrals over arbitrary closed intervals in $\overline{\mathbb{R}}^m$. Perhaps the most interesting aspect of this notion is that it provides a condition that is both *necessary and sufficient.*

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Definition 1 *Let $(f_k) : I \to \mathbb{R}$ $(k \in \mathbb{N})$ be a sequence of functions on a closed interval $I \subseteq \overline{\mathbb{R}}$, and let $f : I \to \mathbb{R}$. We say that (f_k) is γ-convergent to f if for every $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $k \geq m_\varepsilon$ there exists a gauge $\gamma_{\varepsilon,k}$ on I such that if P is any $\gamma_{\varepsilon,k}$-fine partition, then $|S(f_k;P) - S(f;P)| < \varepsilon$.*

Note that the notion of γ-convergence does not require that the functions are integrable, or that the sequence of functions converges at any point. Certainly, the exact significance of the condition is not transparent. However, if the integrals of the functions f_k are to be close to the integral of f, it seems reasonable that the Riemann sums for the f_k should also approximate those of f. It will be observed that the gauges may vary with the index k, and there does not seem to be any uniformity present.

Example 2 (a) If (f_k) is a sequence of functions on a compact interval $I := [a,b]$ that is uniformly convergent to f on I, then given $\varepsilon > 0$ there exists an $m_\varepsilon \in \mathbb{N}$ such that if $k \geq m_\varepsilon$ and $t \in I$ then $|f_k(t) - f(t)| < \varepsilon$. Hence if $P := \{(I_i, t_i)\}_{i=1}^n$ is any tagged partition of I, then

$$|S(f_k;P) - S(f;P)| \leq \sum_{i=1}^{n} |f_k(t_i) - f(t_i)|\, l(I_i) \leq \varepsilon(b-a).$$

Consequently, *on a compact interval*, the uniform convergence of a sequence of functions to a function implies its γ-convergence (with no need to find an appropriate gauge).

(b) Let $I := [0,2]$ and let $g_k(x) := k$ for $x \in (1/k, 1/k + 1/k^2)$ and $g_k(x) := 0$ elsewhere in I, and let $g(x) := 0$ for all $x \in I$. Let the gauge δ_k be defined on I by $\delta_k(x) := \frac{1}{2}\mathrm{dist}(x, \{1/k, 1/k+1/k^2\})$ if $x \in I - \{1/k, 1/k+1/k^2\}$ and $\delta_k(x) := 1$ elsewhere. It is seen that if P is a δ_k-fine partition of I, then $|S(g_k;P) - S(g;P)| \leq 1/k$. Thus the sequence (g_k) is γ-convergent to g. It is to be noted that neither the Monotone or Dominated Convergence Theorems apply to this case. Moreover, the sequence (g_k) is not uniformly integrable on I, since it is readily seen that the sequence of their indefinite integrals is not equicontinuous.

Theorem 3 *Let (f_k) be a sequence of (generalized Riemann) integrable functions on an interval $I \to \mathbb{R}$ and let $f : I \to \mathbb{R}$. Then (f_k) is γ-convergent to f if and only if f is integrable on I and*

$$\int_I f = \lim_{k\to\infty} \int_I f_k. \tag{1}$$

PROOF. ($\Rightarrow$) We will first show that the sequence of integrals $(\int_I f_k)$ is a Cauchy sequence in $\mathbb{R}$. Let $\varepsilon > 0$ and let m_ε be as in Definition 1, so that if $h, k \geq m_\varepsilon$ then there exist gauges $\gamma_{\varepsilon,h}$ and $\gamma_{\varepsilon,k}$ such that if P is a $\gamma_{\varepsilon,h}$-fine partition, then $|S(f_h;P) - S(f;P)| < \varepsilon$, and if P is a $\gamma_{\varepsilon,k}$-fine partition then $|S(f_k;P) - S(f;P)| < \varepsilon$. Further, since f_h and f_k are integrable, there exists gauges $\delta_{\varepsilon,h}$ and $\delta_{\varepsilon,k}$ such that if P is $\delta_{\varepsilon,h}$-fine then $|S(f_h;P) - \int_I f_h| < \varepsilon$, and if P is $\delta_{\varepsilon,k}$-fine then $|S(f_k;P) - \int_I f_k| < \varepsilon$. Now, let $\eta_\varepsilon := \min\{\gamma_{\varepsilon,h}, \gamma_{\varepsilon,k}, \delta_{\varepsilon,h}, \delta_{\varepsilon,k}\}$. Therefore, if P is η_ε-fine, then

$$\left|\int_I f_h - \int_I f_k\right| \leq \left|\int_I f_h - S(f_h;P)\right| + |S(f_h;P) - S(f;P)| + |S(f;P) - S(f_k;P)| + \left|S(f_k;P) - \int_I f_k\right| < 4\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the sequence $(\int_I f_k)$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges to some number $A \in \mathbb{R}$.

We now show that f is integrable and that $\int_I f = A$. For, if $\varepsilon > 0$, let m_ε be as in Definition 1, and let $k \geq m_\varepsilon$ be such that $|\int_I f_k - A| < \varepsilon$. It follows from the γ-convergence of the sequence that there exists a gauge $\gamma_{\varepsilon,k}$ on I such that if P is $\gamma_{\varepsilon,k}$-fine, then $|S(f_k;P) - S(f;P)| < \varepsilon$. Also, from the integrability of f_k, there exists a gauge $\delta_{\varepsilon,k}$ such that if P is $\delta_{\varepsilon,k}$-fine, then $|S(f_k;P) - \int_I f_k| < \varepsilon$. Now let $\eta_{\varepsilon,k} := \min\{\gamma_{\varepsilon,k}, \delta_{\varepsilon,k}\}$. It follows that if P is $\eta_{\varepsilon,k}$-fine, then

$$|S(f;P) - A| \leq |S(f;P) - S(f_k;P)| + \left|S(f_k;P) - \int_I f_k\right| + \left|\int_I f_k - A\right| < 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies that f is integrable on I and that

$$\int_I f = A = \lim_{k\to\infty} \int_I f_k.$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $(\int_I f_k) \to \int_I f$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $k \geq m_\varepsilon$ then $|\int_I f_k - \int_I f| < \varepsilon$. Now let $k \geq m_\varepsilon$ be fixed. Since f_k is integrable there exists a gauge $\delta_{\varepsilon,k}$ such that if P is $\delta_{\varepsilon,k}$-fine, then $|\int_I f_k - S(f_k;P)| < \varepsilon$. Since f is integrable, there exists a gauge $\delta_{\varepsilon,0}$ such that if P is $\delta_{\varepsilon,0}$-fine then $|\int_I f - S(f;P)| < \varepsilon$. Now let $\gamma_{\varepsilon,k} := \min\{\delta_{\varepsilon,0}, \delta_{\varepsilon,k}\}$, so that if P is $\gamma_{\varepsilon,k}$-fine, then

$$\begin{aligned}|S(f_k;P) - S(f;P)| &\leq \left|S(f_k;P) - \int_I f_k\right| + \left|\int_I f_k - \int_I f\right| \\ &\quad + \left|\int_I f - S(f;P)\right| < 3\varepsilon.\end{aligned}$$

Therefore the sequence (f_k) is γ-convergent to f. □

Application 4 Let $f(x) := (-1)^{n-1}/n$ for $x \in [n-1, n)$, $n \in \mathbb{N}$, and let $f(\infty) := 0$. Then f is (generalized Riemann) integrable on the infinite interval $I := [0, \infty]$ and

$$\int_0^\infty f = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Thus f is an example of an integrable function that is not absolutely integrable (and hence is not Lebesgue integrable).

We define $f_k(x) := f(x)$ for $x \in [0, k)$ and $f_k(x) := 0$ for $x \in [k, \infty]$. Since each function f_k is a step function, it is integrable on I and

$$\int_0^\infty f_k = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n}.$$

We claim that the sequence (f_k) is γ-convergent to f on I. For, let $\varepsilon > 0$ be given and let $m_\varepsilon \geq 1/\varepsilon$. If k is any natural number such that $k \geq m_\varepsilon$, we define the gauge $\gamma_{\varepsilon,k}$ on I by

$$\gamma_{\varepsilon,k}(x) := \tfrac{1}{2}\text{dist}(x, \{k, k+1, \cdots\}) \qquad \text{if} \quad x \in [0, \infty) - \{k, k+1, \cdots\},$$

by $\gamma_{\varepsilon,k}(n) := \varepsilon/2^{n+1}$ if $k \leq n \in \mathbb{N}$, and by $\gamma_{\varepsilon,k}(\infty) := \varepsilon$. For the sake of brevity, we only sketch the (rather delicate) argument.

It is not difficult to show that if $P := \{(I_i, t_i)\}_{i=1}^{n}$ is a $\gamma_{\varepsilon,k}$-fine partition of I and if t_i is a tag in P such that $t_i < k$, then $f_k(t_i) = f(t_i)$. Further, $t_n = \infty$, so that $f_k(t_n) = f(t_n) = 0$. Hence we have that

$$S(f; P) - S(f_k; P) = \sum_{t_i = k}^{i=n-1} f(t_i)(x_i - x_{i-1}). \tag{2}$$

From the definition of $\gamma_{\varepsilon,k}$, it is seen that if $p \in \mathbb{N}$ is such that $k \leq p \leq x_{n-1}$, then p is a tag in P and, by splitting the subintervals in P (if necessary), we may assume that p is also a partition point (and hence a tag for two consecutive subintervals). It is readily seen that the first term on the right side of (2) is less than $\varepsilon/2^{k+1}$. Also, a calculation shows that the contribution of the terms having tags in the interval $[p-1, p]$ is within $\varepsilon/2^{p+1}$ of $(-1)^{p-1}/p$. Similarly, the term corresponding to x_{n-1} is seen to be close to $(-1)^{q-1}/q$, where $q = \lfloor x_{n-1} \rfloor$. Thus, the right hand side of (2) is approximately equal to $\sum_{j=k}^{q-1}(-1)^j/(j+1)$. If we use a well-known property of alternating series, we conclude that the expression in (2) can be made arbitrarily small by taking k sufficiently large.

We now show that Theorem 3 can be used to prove the more difficult part of the theorem of Hake that shows that the integral does not admit a "Cauchy extension" by taking limits as the upper limit tends to ∞; see also [1; p.184]. The same type of proof can be used to show that there is no extension as the upper limit approaches a finite point. (It is not claimed that the proofs here are substantially simpler, however.)

Theorem 5 (Hake) *Let $I := [a,\infty]$ and let $f : I \to \mathbb{R}$. If f is (generalized Riemann) integrable on every closed interval $[a,c] \subset [a,\infty)$ and if there exists $A \in \mathbb{R}$ such that*

$$\lim_{c\to\infty} \int_a^c f = A,$$

then f is integrable on I and $\int_a^\infty f = A$.

PROOF. Given $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that if $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$ are such that $m_\varepsilon \le k \le \xi$, then $|\int_k^\xi f| < \varepsilon$. Since f is integrable on $[n-1,n]$, where $k \le n \in \mathbb{N}$, there exists a gauge δ_n on that interval such that if P_n is a δ_n-fine partition of $[n-1,n]$, then $|S(f;P_n) - \int_{n-1}^n f| < \varepsilon/2^n$. We now define the gauge:

$$\gamma_{\varepsilon,k}(x) := \begin{cases} \frac{1}{2}(k-x) & \text{if } x \in [a,k), \\ \min\{\delta_k(k), \delta_{k+1}(k), \varepsilon/(|f(k)|+1)\} & \text{if } x = k, \\ \min\{\delta_n(x), \frac{1}{2}(x-n+1), \frac{1}{2}(n-x)\} & \text{if } x \in (n-1,n)\ (k < n \in \mathbb{N}), \\ \min\{\delta_n(x), \delta_{n+1}(x), \frac{1}{2}\} & \text{if } x = n > k, \\ \varepsilon & \text{if } x = \infty. \end{cases}$$

We define $f_k(x) := f(x)$ if $x \in [a,k)$ and $f_k(x) := 0$ otherwise. Now let $P := \{(I_i,t_i)\}_{i=1}^n$ be a $\gamma_{\varepsilon,k}$-fine partition. It is clear that all of the elements in $\mathbb{N} \cap [k, x_{n-1}]$ are tags in P and, by splitting these terms, we may assume that all of these integers are also endpoints of subintervals in P. Hence we have

$$S(f;P) - S(f_k;P) = \sum_{t_i=k}^{i=n-1} f(t_i)(x_i - x_{i-1}).$$

The contribution to $S(f;P)$ and hence to $S(f;P) - S(f_k;P)$ due to the interval $[x_u,k]$ that has k as right endpoint is $\le |f(k)(k-x_u)| \le \varepsilon$. If $p \ge k+1$, we let $Q_p := P \cap [p-1,p]$; since Q_p is δ_p-fine, we have $|S(f;Q_p) - \int_{p-1}^p f| < \varepsilon/2^p$.

Case 1. If $x_{n-1} = q \in \mathbb{N}$, then

$$\left|\sum_{p=k}^q S(f;Q_p) - \int_k^q f\right| \le \sum_{p=k}^q \frac{\varepsilon}{2^p} < \frac{\varepsilon}{2^{k-1}} \le \varepsilon.$$

Therefore we have

$$|S(f;P)-S(f_k;P)| \le |f(k)|\,(k-x_u)+\left|\sum_{p=k}^{q} S(f;Q_p)\right|+\left|\int_k^q f\right| \le \varepsilon+\varepsilon+\varepsilon = 3\varepsilon.$$

Case 2. If $x_{n-1} \in (q, q+1)$ for some $q \in \mathbb{N}$, then additional terms contribute to $S(f;P)$ from the subintervals in $[q, x_{n-1}]$. But since these subintervals form a subset of a δ_q-fine partition of $[q, q+1]$, it follows from the Saks-Henstock Lemma that this contribution is $\le \varepsilon/2^q < \varepsilon$. Hence, in this case, we have the estimate $|S(f;P) - S(f_k;P)| < 4\varepsilon$.

Consequently we conclude that the sequence (f_k) is γ-convergent to f on $[a, \infty]$, whence the conclusion follows. □

It should be mentioned that Hake's Theorem can also be proved by noting that the sequence (f_k) is uniformly integrable

For additional theorems concerning the convergence of generalized Riemann integrals, see [3] and the papers cited there. In closing, the author would like to thank Professor Gordon for some useful comments.

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