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ON RANGE OF UNIFORMLY ANTISYMMETRIC FUNCTIONS

Abstract

In this note it is proved that the range of uniformly antisymmetric function must have at least four elements. This generalizes the results from [3] and [2] that the range of such function must have at least three elements. The problem whether the range of uniformly antisymmetric function can be finite remains open.

For a linear space $K \subset \mathbb{R}$ over $\mathbb{Q}$ a function $f: K \to \mathbb{R}$ is *uniformly antisymmetric* if for every $x \in K$ there exists $g(x) \in (0,1)$ such that

$$|f(x-h) - f(x+h)| \geq g(x)$$

for every $0 < h < g(x)$, $x \in K$. It is easy to see that $f: K \to \mathbb{Z}$ is uniformly antisymmetric if there exists a function $g: K \to (0,1)$, called a *gage function*, such that

$$f(x-h) \neq f(x+h)$$

for every $0 < h < g(x)$, $h \in K$.

Uniformly antisymmetric functions has been studied by Kostyrko [3], Ciesielski and Larson [2], and Komjáth and Shelah [4]. In [2] it has been proved that there exists an uniformly antisymmetric function $f: \mathbb{R} \to \mathbb{N}$. It has been also shown there that there is no uniformly antisymmetric function $f: K \to \{0,1\}$ for any uncountable linear space $K \subset \mathbb{R}$ over $\mathbb{Q}$, generalizing the result form [3]. The main open problem from [2, Problem 1] is whether there exists a uniformly antisymmetric function $f: \mathbb{R} \to \mathbb{R}$ with finite or bounded range. In this note we will reformulate this problem in terms of n-coloring of

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some infinite graphs on $\mathbb{R}$ and show that there is no uniformly antisymmetric function $f: K \to \{0,1,2\}$ for any uncountable linear space $K \subset \mathbb{R}$ over $\mathbb{Q}$.

We need some definitions and facts. Let $K \subset \mathbb{R}$ be a linear space over $\mathbb{Q}$. For a gage function $g: K \to (0,1)$ define a graph $G_g = (K, E_g)$ by considering K as its set of vertices and defining the set E_g of its edges as the set of all unordered pairs $\{a,b\}$ from K such that $g(x) \geq |x-a| = |x-b|$ for $x = (a+b)/2$. Notice that for $B \subset \mathbb{Z}$ there exists a uniformly antisymmetric function $f: K \to B$ with a gage function g if and only if f is a coloring of the graph G_g such that no two vertices connected by an edge have the same color. In other words, there exists a uniformly antisymmetric function $f: K \to \{0,1,\ldots,n-1\}$ with gage g if and only if the graph G_g is n-colorable.

Since graph G_g is n-colorable if and only if every its finite subgraph is n-colorable (see [1]) we conclude the following theorem.

Theorem 1 *For any linear space $K \subset \mathbb{R}$ over $\mathbb{Q}$ and any natural number n there exists a uniformly antisymmetric function $f: K \to \{1,2,\ldots,n\}$ if and only if there exists a function $g: K \to (0,1)$ such that every finite subgraph of G_g is n-colorable.*

In particular, there is no uniformly antisymmetric function $f: K \to \mathbb{R}$ with finite range if and only if for every $g: K \to (0,1)$ and every natural number n the graph G_g contains a finite subgraph which is not n-colorable. □

Now, we are ready to prove the following theorem.

Theorem 2 *Let K be an uncountable linear space over $\mathbb{Q}$. If $f: K \to \{1,2,3\}$ then f is not uniformly antisymmetric function.*

Proof. Choose arbitrary $g: K \to (0,1)$. By Theorem 1 it is enough to show that G_g contains finite subgraph which is not 3-colorable. To this order we will show that G_g contains K_4, i.e., graph with 4 vertices and all possible edges.

We denote vertices of K_4 by A, B, C and D. We will think of this K_4 as on 3-dimensional tetrahedron with base formed by vertices A, B and C. (See figure.) Let a, b and c denote the mid points of intervals (edges) BC, AC and AB, respectively. (Thus, in triangle ABC, point a is a center of the side opposite to A, etc.) Also, centers of sides AD, BD and CD are denoted by a', b' and c', respectively.

The algebraic relation between these points is given by equations

$$A+B=2c, \quad B+C=2a, \quad A+C=2b, \tag{1}$$

$$A+D=2a', \quad B+D=2b', \quad C+D=2c'. \tag{2}$$

Solving (1) we obtain

$$A=-a+b+c, \quad B=a-b+c, \quad C=a+b-c. \tag{3}$$

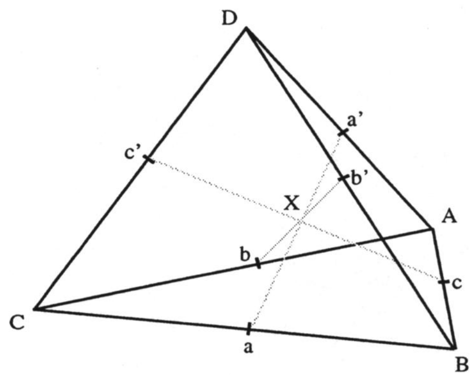

Figure 1: K_4 embedded into G_g.

In particular, to insure that graph generated by triangle ABC is in G_g it is enough to make sure that there exists $\varepsilon > 0$ such that

$$|a - B| = |b - c| < \varepsilon, \ |b - C| = |a - c| < \varepsilon, \ |c - A| = |a - b| < \varepsilon \tag{4}$$

and

$$\varepsilon < g(a), \quad \varepsilon < g(b), \quad \varepsilon < g(c). \tag{5}$$

Notice also, that adding each of the equations (1) to an appropriate equation from (2) we obtain

$$2(a + a') = 2(b + b') = 2(c + c') = A + B + C + D. \tag{6}$$

So, intervals $[a, a']$, $[b, b']$ and $[c, c']$ have a common center. Denote this common center by X. Then,

$$a' = 2X - a, \quad b' = 2X - b, \quad c' = 2X - c. \tag{7}$$

Let δ denote the diameter of the set $\{a, b, c, a', b', c'\}$ and notice that, by (2) and (3),

$$|a' - D| = |A - a'| = |-a + b + c - a'| \leq |-a + b| + |c - a'| \leq 2\delta.$$

Similarly we can show that $|b' - D| \leq 2\delta$ and $|c' - D| \leq 2\delta$. Thus, in order to insure that edges AD, BD and CD are in G_g it is enough to choose $ABCD$ such that

$$g(a') > 2\delta, \quad g(b') > 2\delta, \quad g(c') > 2\delta. \tag{8}$$

Now, we are ready to make the choice of our K_4. First, choose $d \in (0, 1/4)$ and an uncountable $S \subset K$ such that $g[S] \subset (4d, 1)$. Pick $X \in K$ such that $S \cap (X-d, X+d)$ is uncountable and define $T = \{2X - s: s \in S \cap (X-d, X+d)\}$. Thus, $T \subset (X-d, X+d)$ is uncountable. Choose uncountable subset $T_1 \subset T$ and $\varepsilon > 0$ such that $g[T_1] \subset (\varepsilon, 1)$. We can pick $a, b, c \in T_1$ such that $a < b < c < a+\varepsilon$. Now, from points a, b, c and X we can reconstruct A, B, C, D, a', b', c' as described above.

Edges AB, BC and AC are in G_g, since a, b, c satisfy (4) and (5).

To see that edges AD, BD and CD are in G_g notice that $a', b', c' \in S$, i.e., $g(a') > 4d$, $g(b') > 4d$ and $g(c') > 4d$. To finish the proof it is enough to notice that $a, b, c, a', b', c' \in (X-d, X+d)$, i.e., that $\delta < 2d$, since this implies (8). □

The following questions seems to be intriguing in light of the previous theorem.

Problem 1 *Can we embed K_5 into G_g for every $g: \mathbb{R} \to \mathbb{R}$?*

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