Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 459–464

Dr. Harry Poppe, Universität Rostock, Fachbereich Mathematik, PF 999, 0-2500 Rostock 1, Germany

CONVERGENCE OF EVENLY CONTINUOUS NETS IN GENERAL FUNCTION SPACES

Abstract

For topological spaces X, Y let Y^X, $C(X,Y)$ denote the sets of all functions from X to Y and of all continuous functions from X to Y respectively. We consider nets of functions from these spaces which are evenly continuous or pointwise equicontinuous, and for such nets the relationship between pointwise convergence and topological convergence is studied. We find that in Y^X for an evenly continuous net pointwise convergence (to a function from Y^X) implies topological convergence. Conversely, if Y is an uniform space and (f_i) a pointwise equicontinuous net then $\Gamma f \imath \liminf \Gamma f_i$ implies the pointwise convergence of (f_i) to f. By an example is shown that in the second assertion the equicontinuity of (f_i) cannot be replaced by even continuity. As corollaries of our results we get some results of T. Neubrunn and Ľ. Holà [6] and of G. Beer [1] respectively. Moreover, a relationship to the lower semi-finite graph-topology is established.

1. Introduction

We consider topological spaces X, Y; by Y^X, $C(X,Y)$ we denote the sets of all functions from X to Y and of all continuous functions from X to Y respectively. Let $(f_i)_{i \in I}$ be an arbitrary net from Y^X and we assume that (f_i) is evenly continuous (in the sense of J. L. Kelley and A. P. Morse see [3], [10]).

For such a net we study in Y^X the relationship between the convergence with respect to the topology of pointwise convergence τ_p and topological convergence, which means the Hausdorff convergence of the graphs of the functions f_i. At this our main goal in this note is to show by an example that

Mathematical Reviews subject classification: 54 C 35
Received by the editors February 3, 1992

the assumption that (f_i) is evenly continuous is too weak for proving that $\Gamma f \imath \liminf \Gamma f_i$ implies the pointwise convergence of (f_i) to f, where (f_i) is a net from Y^X, $f \in Y^X$ and $\liminf \Gamma f_i$ denotes the closed limit inferior of the net of graphs (Γf_i).

To prove this implication instead of this assumption, we must assume that Y is a uniform space and (f_i) is a pointwise equicontinuous net. As corollaries of our results we get some results of T. Neubrunn and Ľ. Holà [6] and of G. Beer [1] respectively.

Moreover, we get as a corollary for nets in Y^X a relationship of τ_p-convergence and convergence with respect to the graph-topology τ_1 (see [9] for a definition of τ_1).

For sake of completeness and for explaining the notation we will define the (well-known) notion of topological convergence for a net of subsets of a topological space (see also [2], [4], [10]).

(1) Let X be a topological space and let $(A_i)_{i\in(I,\leq)}$ be a net of subsets of X

a) The set $\liminf A_i := \{x \in X :$ for each neighborhood U of x there exists $i = i_U \in I$ such that $i \geq i_U$ implies $A_i \cap U \neq \phi\}$ is called the closed limit inferior or the closed topological lower limit of the net (A_i).

b) The set $\limsup A_i := \{x \in X :$ for each neighborhood U of x there exists a subnet $(B_k)_{k\in K_U}$ of $(A_i)_{i\in I}$ such that $B_k \cap U \neq \phi$ for each $k \in K_U\}$ is called the closed limit superior or the closed topological upper limit of the net (A_i).

c) We say that (A_i) converges topologically to the set $A \imath X$ and A is the topological limit or the closed topological limit of the net (A_i), $A = t - \lim A_i$, iff $\liminf A_i = \limsup A_i = A$.

Remark 1 $\liminf A_i \imath \limsup A_i$ *obviously holds, and thus* $A = t - \lim A_i$ *is equivalent to* $A \imath \liminf A_i \imath \limsup A_i \imath A$.

2. Pointwise and topological convergence

In the first part of the following theorem we use the general assumption that the net (f_i) is evenly continuous, but in the second part we must assume that Y is a uniform space and (f_i) is pointwise equicontinuous. In the next section we will show by an example that this stronger assumption cannot be dropped, as was announced in the introduction.

We will use the well-known notion $f_i \xrightarrow{c} f$ of continuous convergence of a net (f_i) from Y^X to $f \in Y^X$. Especially a splitting topology τ for Y^X can be characterized by: for each net (f_i) from Y^X, $f \in Y^X$ $f_i \xrightarrow{c} f$ implies $f_i \xrightarrow{\tau} f$ (see [10]).

(2) Theorem:

1. Let X, Y be topological spaces, (f_i) an evenly continuous net from Y^X and let be $f \in Y^X$. Then $f_i \xrightarrow{\tau_p} f$ implies $t - \lim \Gamma f_i = \Gamma f$.

2. Let be X a topological space and (Y, α) an uniform space, where α denotes the set of all entourages of the uniform structure. Let (f_i) be a pointwise equicontinuous net from Y^X and let be $f \in Y^X$. Then $\Gamma f \subset \liminf \Gamma f_i$ implies $f_i \xrightarrow{\tau_p} f$.

3. Under the assumptions of 2., we find that $f_i \xrightarrow{p} f$ is equivalent to $\Gamma f \subset \liminf \Gamma f_i$.

PROOF.

1. If (f_i) is evenly continuous and (f_i) converges pointwise to f then (f_i) converges continuously to f too, which implies by a theorem of O. Frink that $t - \lim \Gamma f_i = \Gamma f$ holds (see [7], [10]).

2. This proof we get by straightforward arguments: let $x \in X$ and $V(f(x))$ be an arbitrary neighborhood of $f(x)$, where $V \in \alpha$; we find a symmetric entourage $W \in \alpha$ such that $W \circ W \subset V$. Since (f_i) is pointwise equicontinuous there exists a neighborhood U of x such that $z \in U$ and $i \in I$ imply $(f_i(z),\ f_i(x)) \in W$. Now we have $(x, f(x)) \in \Gamma f$ and hence $(x, f(x) \in \liminf \Gamma f_i$ too implying that we find $i_0 \in I$ and for each $i \geq i_0$ points $(z_i, w_i) \in (U \times W(f(x)) \cap \Gamma f_i$; since $z_i \in U$, $w_i = f_i(z_i) \in W(f(x))$ we have $(f_i(z_i),\ f_i(x)) \in W$ and $(f_i(z_i),\ f(x)) \in W$ implying $f_i(x) \in V(f(x))$.

3. If (f_i) is pointwise equicontinuous, then (f_i) is evenly continuous also [3] and hence assertion 3. follows from 1. and 2.

Remark 2 *If (f_i) is evenly continuous and Y is regular, then we have $\{f_i\} \subset C(X, Y)$ and the τ_p-closure is also evenly continuous and hence is contained in $C(X, Y)$. Moreover if Y is regular and $f_i \xrightarrow{c} f$, then f is continuous (see [10]). Hence if Y is regular (or Y is an uniform space), we can work in $C(X, Y)$.*

We want to give some comments on the assertions of the theorem.

a) Assertion 1. clearly holds if (f_i) is not evenly continuous but is pointwise equicontinuous. In this form assertion 1. generalizes a result of G. Beer [1], who proved it for the case of sequences in $C(X, Y)$, where X, Y are metric spaces.

b) Corollary: Let us consider the assumption of assertion 2; let τ be a splitting topology for Y^X. Then $\Gamma f \imath \liminf \Gamma f_i$ implies $f_i \xrightarrow{\tau} f$.

PROOF. By Theorem (2), 2. $\Gamma f \imath \liminf \Gamma f_i$ implies $f_i \xrightarrow{\tau_p} f$; since a pointwise equicontinuous family is evenly continuous the pointwise convergence of (f_i) to f implies the continuous convergence as was pointed out in the proof of assertion 1. But from $f_i \xrightarrow{c} f$ we obtain $f_i \xrightarrow{\tau} f$ too.

Remark 3 *For instance for τ we can use the compact-open topology in the corollary. For the compact-open topology in $C(X,Y)$ and sequences from $C(X,Y)$, where X,Y are metric spaces the assertion of the corollary was proved by T. Neubrunn and Ľ. Holà [6].*

3. An example

Let be $X = [0,1]$, $Y = [0,+\infty]$, where for both spaces we consider the Euclidean topology. We want to construct a sequence (f_n) and a function f from Y^X (even we can choose (f_n), f from $C(X,Y)$) such that hold:

1. $\Gamma_f \imath \liminf \Gamma f_n$
2. (f_n) is evenly continuous
3. (f_n) does not converge pointwise to f.

Let be for $n = 1,2,\ldots, f_n : f_n(x) = \begin{cases} 0, & x \in [\frac{1}{n},1] \\ -n^2x+n, & x \in [0,\frac{1}{n}) \end{cases}$ and let f be the zerofunction on X. Clearly f_n, $f \in C(X,Y)$. Since $(f_n(0)) = (n)$, assertion 3. holds, and using the definiton of lim inf it is not hard to see that assertion 1. holds, too. We will prove assertion 2. using the classical definition of even continuity [3]: for each $(x,y) \in X \times Y$ and each neighborhood V of y we must find neighborhoods U of x and W of y such that for each n, $f_n(x) \in W$ implies $f_n(U) \imath V$.

We have the possibilities 1. $y = 0$ and 2. $y > 0$. If $y = 0$ and $x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ then we can find such a small neighborhood W of y such that $f_k(x) = f_k(\frac{1}{n}) \in W$ holds only if $k \geq n$. But then choosing U in such a way that $U \imath (\frac{1}{2}\frac{1}{n+1}, \frac{1}{2}\frac{1}{n-1})$, $n > 1$ (or $U \imath (\frac{1}{4}, 1]$, if $n = 1$) and $f_n(U) \imath V$, since $f_n(\frac{1}{n}) = 0 = y$ and f_n is continuous, we find that $f_n(U) \imath V$ and $f_k(U) = \{0\} \imath V$ for $k > n$ hold.

If $y = 0$ and $x \notin \{\frac{1}{n} : n = 1,2,\ldots\}$ we should distinguish between $x = 0$ and $0 < x \leq 1$. If $x = 0$ we can choose $W = [0, \frac{1}{2})$ and an arbitrary U yielding $f_k(0) \notin W$ for each k. Hence the implication $n \in \{1,2,\ldots\}$ and $f_n(0) \in W$

implies $f_n(U)\imath V$ is true. If $0 < x \leq 1$ let be $\frac{1}{n+1} < x < \frac{1}{n}$ and let be W be a small neighborhood of y such that $f_k(x) \in W$ holds only for $k \geq n+1$. Choosing U in such a way that $U\imath(\frac{1}{n+1}, \frac{1}{n})$ holds, we find: $f_k(x) \in W$ implies $f_k(U) = \{0\}\imath V$.

Now we consider $y > 0$ and we remark at first that as is easily seen in $\{(x,y)|0 < x \leq 1, 0 < y\}$ only two graphs of functions from the sequence (f_n) intersect in a common point. If Γf denotes the graph of a function f we have three cases:

a) $(x,y) \notin \Gamma f_k$ for each k,

b) $(x,y) \in \Gamma f_k$ for one k, and

c) $(x,y) \in \Gamma f_k$ for two numbers k.

In each case we can then find a small neighborhood W of y such that for a) $f_k(x) \notin W$ for each k, for b) $f_k(x) \in W$ only for one k and for c) $f_k(x) \in W$ only for two distinct numbers k. For b) and c) we have $y = f_k(x)$ and since each function f_k is continuous for each of the three cases we find U such that the implication $f_k(x) \in W$ implies $f_k(U)\imath V$ is true.

Remark 4 *Clearly the constructed family (f_n) is not pointwise equicontinuous. For another example of such a family of functions see [10].*

4. An application to the topology τ_1

The topology τ_1 for Y^X is the lower semi-finite topology in the sense of Michael [5] for the set of all graphs $\{\Gamma f : f \in Y^X\}$. For a net (f_i) and f from Y^X in [9] was shown:

1. $f_i \xrightarrow{\tau_p} f$ implies $f_i \xrightarrow{\tau_1} f$
2. $f_i \xrightarrow{\tau_1} f$ is equivalent to $\Gamma f \imath \liminf \Gamma f_i$.

Combining this with theorem (2), 2. we get:

(3) Let X be a topological space, let Y be a topological or a uniform space respectively. Let (f_i) be a net in Y^X and let be $f \in Y^X$. Then we have:

a) $f_i \xrightarrow{\tau_p} f$ implies $f_i \xrightarrow{\tau_1} f$

b) If (f_i) is equicontinuous then $f_i \xrightarrow{\tau_1} f$ implies $f_i \xrightarrow{\tau_p} f$.

Proof of b): $f_i \xrightarrow{\tau_1} f$ implies $\Gamma f \imath \liminf \Gamma f_i$ which yields by (2), 2. $f_i \xrightarrow{\tau_p} f$.

References

[1] G. Beer, *More on convergence of continuous functions and topological convergence of sets*, Canad. Math. Bull. **28** (1985), 52–59.

[2] W. Gähler, "Grundstrukturen der Analysis" II, Berlin, 1978.

[3] J.L. Kelley, "General Topology", Princeton, N.J., 1955.

[4] E. Klein, A. Thomson, "Theory of Correspondences", Wiley, New York, 1984.

[5] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[6] T. Neubrunn, Ľ. Holà, *On almost uniform convergence and convergence in Hausdorff metric*, Radovi Matematicki **4** (1988), 193–205.

[7] H. Poppe, *Einige Bemerkungen über den Raum der abgeschlossenen Mengen*, Fund. Math. **59** (1966), 159–169.

[8] H. Poppe, *Über Graphentopologien für Abbildungsräume I*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **15** (1967), 71–80.

[9] H. Poppe, *Über Graphentopologien für Abbildungsräume II*, Math. Nachr. **38** (1969), 89–96.

[10] H. Poppe, "Compactness in General Function Spaces", Deutscher Verlag der Wissenschaften, Berlin, 1974.