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H. Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, Alabama, 35487

R.G. Gibson, Department of Mathematics and Computer Science, Columbus College, Columbus, Georgia, 31993

F. Roush, Department of Mathematics, Alabama State University, Montgomery, Alabama 36101.

Extendable Functions and Almost Continuous Functions with a Perfect Road

The main results obtained in this paper are (1) if $g : I^2 \to I$ is a connectivity function and z is an interior point of the range of g not equal to 0 or 1, then $g^{-1}(z)$ separates I^2; and (2) if $f : I \to I$ is an extendable function, then f has the strong Cantor intermediate value property (SCIVP). Examples are given that show that (3) the first main result is false for Darboux functions and for almost continuous functions and (4) there exists an almost continuous function $f : I \to I$ with a perfect road at each point that does not have the SCIVP. Hence f is not extendable. In this paper, $I = [0, 1]$ and $\bar{A}$ is the closure of the set A.

Gibson and Roush, in [3], defined the Cantor intermediate valve property (CIVP) and conjectured that if $f : I \to I$ is an extendable function, then f has the CIVP. In this paper, we define the strong Cantor intermediate valve property (SCIVP) and then prove the conjecture by showing that if $f : I \to I$ is an extendable function, then f has the SCIVP (The SCIVP implies the CIVP).

Stallings defined almost continuous functions in [14] and proved that if $f : I \to I$ is an extendable function, then f is almost continuous. In [4], Gibson and Roush (1) defined the weak Cantor intermediate valve property (WCIVP), (2) constructed an almost continuous function $f : I \to I$ that does not have the WCIVP, and (3) proved that if $f : I \to I$ is extendable, then f has the WCIVP (The CIVP implies the WCIVP). Hence almost continuity does not imply extendability.

In [5], Gibson and Roush gave an example of an almost continuous function $f : I \to I$ that does not have a perfect road and proved that if f is an extendable function, then f has a perfect road. They asked in [5] whether there exists an almost continuous function $f : I \to I$ that has a perfect road and is not an extendable function. In this paper, we give an affirmative answer to this question by constructing an almost continuous function $f : I \to I$ that has a perfect road but does not have the CIVP. The example is dense in I^2 and is not Borel measurable because it is not Marczewski measurable [12].

We now give the definition of the properties and functions needed in this paper.

Let X and Y be topological spaces and let $f : X \to Y$. Then:

D.: f is a Darboux function if $f(C)$ is connected whenever C is connected in X.

Conn.: f is connectivity function if the graph of f restricted to C, denoted $f|C$, is connected in $X \times Y$ whenever C is connected in X.

A.C.: f is an almost continuous function if whenever $U \subset X \times Y$ is an open set containing the graph of f, then U contains the graph of a continuous function $g : X \to Y$.

Ext.: f is an extendable function if there exists a connectivity function $g : X \times I \to Y$ such that $f(x) = g(x, 0)$ for any $x \in X$.

P.C.: f is peripherally continuous if for each $x \in X$ and for each pair of open sets U and V such that $x \in U$ and $f(x) \in V$, there exists an open set W containing x such that $\bar{W} \subset U$ and $f(bd(W)) \subset V$ where $bd =$ boundary.

Let $f : [a, b] \to R$ be a function. Then:

P.R.: f has a perfect road if for each x in $[a, b]$, there exists a perfect set P having x as a bilateral limit point such that $f|P$ is continuous at x. If x is an endpoint, then the bilateral condition is replaced with a unilateral condition.

CIVP.: f has the CIVP if for p and q in $[a, b]$ such that $p \neq q$ and $f(p) \neq f(q)$ and for any Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$.

WCIVP.: f has the WCIVP if for p and q in $[a, b]$ such that $p \neq q$ and $f(p) \neq f(q)$, there exists a Cantor set C between p and q such that $f(C)$ is between $f(p)$ and $f(q)$.

SCIVP.: f has the SCIVP if for p and q in $[a, b]$ such that $p \neq q$ and $f(p) \neq f(q)$ and for any Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$ and $f|C$ is continuous.

Let $f : I^n \to I$, $n \geq 1$. If $n > 1$ and f is a connectivity function, then f is almost continuous [14]. However, if $n = 1$ and f is almost continuous, then f is a connectivity function. For functions $f : I^n \to I$, $n > 1$, the set W in the definition of peripherally continuous functions can be selected so that W and $bd(W)$ are connected [14]. Also for functions $f : I^n \to I$, $n > 1$, Conn. = P.C. [8], [17].

If f is a real-valued function defined on an interval $[a, b]$, then f is a connectivity function if and only if the entire graph of f is connected. Now f is a Darboux function if and only if f has the intermediate value property. Also if f is a connectivity function, then f is a Darboux function; and if f is a Darboux function, then f is a peripherally continuous function. However, for real-valued functions defined on an interval that are of Baire Class 1, Ext. = A.C. = Conn. = D. = P.C. = P.R. [1].

Blocking set: A blocking set H of a function $f : X \to Y$ is a closed subset of $X \times Y$ that contains no point of f but intersects every continuous function $h : X \to Y$.

If $f : I \to I$ is not almost continuous, then there exists a minimal blocking set H of f and the projection $\pi_1(H)$ of H into the x-axis is a non-degenerate connected set [10].

We now give the results stated in the introduction. For convenience, we make the following definition.

Leaf: If $h : X \to Y$ is a function and $y \in Y$, then a component of $h^{-1}(y)$ is called a leaf. If B is a subset of Y, then a leaf L of $h^{-1}(B)$ means that there exists a $y \in B$ such that L is a leaf of $h^{-1}(y)$.

Theorem 1. *Let $g : I^2 \to I$ be a connectivity function. Then for each $\varepsilon > 0$, the restriction of g to the union A of all leaves of $g^{-1}(I)$ which have diameter greater than or equal to ε is continuous.*

Proof. Let $\{L_\alpha\}$ be the collection of all leaves of $g^{-1}(I)$ which have diameter greater than or equal to ε. Then $A = \bigcup_\alpha L_\alpha$.

Select any $x \in A$ and let x_n be a sequence in A which converges to x. If x_n converges to x along a leaf, then x is in that leaf and $g(x_n) = g(x)$ for each n [14].

Without loss of generality, we may assume that each x_n is in a different leaf L_n, and suppose that $g(x_n)$ does not converge to $g(x)$. We may also assume, without loss of generality, that $g(x)$ is not a cluster point of $g(x_n)$.

Let N be a neighborhood of x with diameter $< \frac{1}{2}\varepsilon$ such that the complement,

$\tilde{N}$, is connected. Then

$$B = (\bigcup_{n=1}^{\infty} L_n) \cup \tilde{N} \cup \{x\}$$

is connected, but $(x, g(x))$ is an isolated point of the graph of $g|B$. This is a contradiction. Hence $g(x_n)$ converges to $g(x)$ and $g|A$ is continuous at x. Consequently, $g|A$ is continuous.

Theorem 2. *If $g : I^2 \to I$ is a connectivity function and A is the same as in theorem 1, then $g|\bar{A}$ is continuous.*

Proof: Select any $x \in \bar{A}$ and let $x_n \in \bar{A}$ be a sequence such that x_n converges to x.

Suppose each $x_n \in A$. If x_n converges to x along a leaf, then $x \in A$ and $g(x_n) = g(x)$ for each n. Without loss of generality, we may assume that each x_n is in a different leaf L_n. Suppose $g(x_n)$ does not converge to $g(x)$. We may assume that $g(x)$ is not a cluster point of $g(x_n)$. Let N be a neighborhood of x with diameter $< \frac{1}{2}\varepsilon$ such that $\tilde{N}$ is connected. Then

$$B = (\bigcup_{n=1}^{\infty} L_n) \cup \tilde{N} \cup \{x\}$$

is connected, but $(x, g(x))$ is an isolated point of the graph of $g|B$. This is a contradiction. So $g(x_n)$ converges to $g(x)$.

Now, without loss of generality, we may assume that each $x_n \in \bar{A} - A$. We show that for any $\delta > 0$ there exists a positive integer M such that if $n \geq M$, then $|g(x_n) - g(x)| < \delta$. Let $\delta > 0$ be given. Construct a sequence $y_n \in A$ such that y_n converges to x and $|g(x_n) - g(y_n)| < \frac{1}{2}\delta$. To do this, select $y_n \in A$ such that distance $(x_n, y_n) < 1/n$. From above, $g(y_n)$ converges to $g(x)$. Thus there exists a positive integer M such that if $n \geq M$, then $|g(y_n) - g(x)| < \frac{1}{2}\delta$. So if $n \geq M$, then $|g(x_n) - g(x)| \leq |g(x_n) - g(y_n)| + |g(y_n) - g(x)| < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$. So $g(x_n)$ converges to $g(x)$ and $g|\bar{A}$ is continuous at x. Hence $g|\bar{A}$ is continuous.

Theorem 3. *If $g : I^2 \to I$ is a connectivity function and A is the same as in Theorem 1, then $\bar{A} = A$.*

Proof. Since $A = \bigcup_\alpha L_\alpha$, we need to show that if x_n is in L_n for a sequence $\{L_n\}$ in $\{L_\alpha\}$ and x_n converges to x, then x is in a leaf with diameter $\geq \varepsilon$.

Since the space of continua in I^2 is compact, we may choose a subsequence $\{L_{n(m)}\}$ of $\{L_n\}$ which converges. The limit is a continuum L and contains x. Also the diameter of L is $\geq \varepsilon$.

Let $a \in L$ and let $a_m \in L_{n(m)}$ be a sequence such that a_m converges to a. Then $g(a_m) = g(x_{n(m)})$. Since $g(x_{n(m)})$ converges to $g(x)$ and $g(a_m)$ converges to $g(a)$, $g(x) = g(a)$. Thus L is contained in a leaf with diameter $\geq \varepsilon$.

Before continuing, we make the following definitions.

Quasi-component: If A is any set and p is any point of A, then by a quasi-component of A containing p is meant the set consisting of p together with all points x of A such that A is not separated between p and x; i.e., there exists no separation $A = A_p \cup A_x$ where $\bar{A}_p \cap A_x = A_p \cap \bar{A}_x = \emptyset$, $p \in A_p$, and $x \in A_x$. For a reference see [16].

The quasi-components of any set A are disjoint and closed in A. A quasi-component of A may not be connected, but each component of A is contained in a single quasi-component. In general, the quasi-components of a set may be different from the components of this set.

Semi-closed: K is a semi-closed subset of a metric space X if and only if each component of K is closed in X and any convergent sequence of components of K whose limit set intersects $X - K$ converges to a single point.

Theorem 4. *If $g : I^2 \to I$ is a connectivity function and z is an interior point of the range of g not equal to 0 or 1, then any point of $H = g^{-1}([0, z))$ and any point of $K = g^{-1}((z, 1])$ lie in different quasi-components of $I^2 - g^{-1}(z)$.*

Proof. Part of the proof of this theorem is contained in the proof of theorem 2.1 of Hunt [9]. For completeness we give that part here.

It follows from theorem 3.1 of Hagan [7] that $g^{-1}(z)$ is a semi-closed subset of I^2.

From 5.2 on page 132 of Whyburn [16], we have that the components of the semi-closed set $g^{-1}(z)$ and the single points of $I^2 - g^{-1}(z)$ form an upper semi-continuous decomposition S of I^2.

From the theorem on page 127 of [16], the natural projection $\pi : I^2 \to \pi(I^2) = S$ is a monotone function onto the decomposition space S. Since π is continuous, by 2.21 on page 138 of [16], $\pi(I^2) = S$ is a unicoherent Peano continuum. By 2.2 on page 138 of [16], $\pi(g^{-1}(z))$ is totally disconnected in $\pi(I^2)$. By lemma 1 of Cornette and Girolo [2], the quasi-components of $\pi(I^2 - g^{-1}(z)) = \pi(I^2) - \pi(g^{-1}(z))$ are connected. From 5.3 on page 132 of [16],

$$\pi|(I^2 - g^{-1}(z)) : (I^2 - g^{-1}(z)) \to \pi(I^2 - g^{-1}(z))$$

is a homeomorphism. So the quasi-components of $I^2 - g^{-1}(z)$ are connected.

Let $a \in H$ and $b \in K$. Then $g(a) \in [0, z)$ and $g(b) \in (z, 1]$. If a and b were in the same quasi-component Q of $I^2 - g^{-1}(z) = H \cup K$, then Q is connected and $z \in g(Q)$. This is a contradiction and the theorem is proved.

Theorem 5. *If $g : I^2 \to I$ is a connectivity function and z is an interior point of the range of g not equal to 0 or 1, then $g^{-1}(z)$ separates I^2.*

The conclusion of this theorem is not true for Darboux functions and for almost continuous functions. Consider the following example.

Example 1: Define $h : [-1, 1] \times [0, 1] \to [-1, 1]$ by

$$\begin{aligned} h(x, y) &= \sin(1/y) \text{ if } y > 0, \text{ and} \\ h(x, 0) &= x \text{ otherwise} \end{aligned}$$

Now h is a Darboux function and an almost continuous function but not a connectivity function.

Let A be the set consisting of the line segment joining the points $(0, 0, 0)$ and $(0, 1, 0)$, the line segments that are the intersection of the graph of $h(x, y) = \sin(1/y)$ and the xy-plane, and the point $(1, 0, 0)$. This set A is connected, but the graph of the $h|A$ is not connected.

Now 0 separates 1 and -1. But $h^{-1}(0)$ does not separate each point of $h^{-1}(1)$ from each point of $h^{-1}(-1)$, since $(1, 0)$ and $(-1, 0)$ are in the same quasi-component of $\widetilde{h^{-1}(0)}$.

Theorem 6. *If $g : I^2 \to I$ is an extension of $f : I \to I$ and g is a connectivity function, then f has the SCIVP.*

Proof. Let $a, b \in I$ such that $a < b$ and $f(a) = g(a, 0) \neq g(b, 0) = f(b)$. Let K be a Cantor set between $g(a, 0)$ and $g(b, 0)$. Select any $z_1 \in K$ such that z_1 is not an endpoint of an interval removed. Then $g^{-1}(z_1)$ separates $(a, 0)$ from $(b, 0)$ in I^2. So there exists a leaf L_1 of $g^{-1}(z_1)$ such that L_1 separates $(a, 0)$ from $(b, 0)$ in I^2 according to theorem 4.12 on page 51 of Wilder [15]. Let $(x_1, 0)$ be a point of $L_1 \cap (I \times \{0\})$ such that $a < x_1 < b$.

Select $z_2 \in K$ such that z_2 is not an endpoint of an interval removed and z_2 is between z_1 and $g(b, 0)$. Then $g^{-1}(z_2)$ separates $(x_1, 0)$ from $(b, 0)$ in I^2. So there exists a leaf L_2 of $g^{-1}(z_2)$ such that L_2 separates $(x_1, 0)$ from $(b, 0)$. Let $(x_2, 0)$ be a point of $L_2 \cap (I \times \{0\})$ such that $x_1 < x_2 < b$.

Let $K^* = K \cap [z_1, z_2]$ or $K^* = K \cap [z_2, z_1]$. Then K^* is a Cantor set such that $K^* \subset K$.

Select any $\varepsilon > 0$ such that $\varepsilon < \min\{\text{diam}(L_1), \text{diam}(L_2), 1\}$. Then the union F of all leaves of $g^{-1}(K^*)$ with diameter $\geq \varepsilon$ is closed and g restricted to F is continuous. Now there exist c-many points of $g^{-1}(K^*)$ in $I \times \{0\}$ between $(x_1, 0)$ and $(x_2, 0)$. So $F \cap ([x_1, x_2] \times \{0\})$ is closed and contains c-many points. Thus it contains a Cantor set $C \times \{0\}$.

Since $g|F$ is continuous, $g|(C \times \{0\})$ is continuous and $g(C \times \{0\}) \subset K$. Since $C \subset [x_1, x_2] \subset I$, $g|(C \times \{0\}) = f|C$. Therefore $f(C) \subset K$ and $f|C$ is continuous.

Example 2: There exists a connectivity function $g : I^2 \to I$ such that for some $x_0 \in I$ the restriction $h = g|(I \times \{x_0\})$ is a connectivity function onto I whose graph is dense in $I \times \{x_0\} \times I$.

From the example in [6] of a connectivity function $g : I^2 \to I$, there exists a dense G_δ-subset G of I^2 on which the values of g can be chosen to be either 0 or 1. From [13], there exists an $x_0 \in I$ such that $G_0 = G \cap (I \times \{x_0\})$ is a dense G_δ-subset of $I \times \{x_0\}$.

Let Z be a Bernstein subset of I; see theorem 1, page 514, of [11]. Then each of $A = G_0 \cap (Z \times \{x_0\})$ and $B = G_0 \cap ((I \times \{x_0\}) - (Z \times \{x_0\}))$ is dense in $I \times \{x_0\}$. Redefine g in the example in [6] so that $g(A) = 0$ and $g(B) = 1$ keeping g fixed otherwise. Now $h = g|(I \times \{x_0\})$ is a connectivity function dense in $I \times \{x_0\} \times I$ and onto I. To prove these claims, consider the following.

Let $(a, b) \times \{x_0\} \times (r, s)$ be an open square in $I \times \{x_0\} \times I$. Then $(a, b) \times \{x_0\}$ is an open interval in $I \times \{x_0\}$. So $G_0 \cap ((a, b) \times \{x_0\})$ is a dense G_δ-subset of $(a, b) \times \{x_0\}$, and hence it contains a Cantor set, page 53 of [16]. Thus there exist points $x_1, x_2 \in (a, b)$ such that $(x_1, x_0) \in A$ and $(x_2, x_0) \in B$. Hence A and B are dense in $I \times \{x_0\}$.

Now $g(x_1, x_0) = 0$ and $g(x_2, x_0) = 1$. Select any $y \in (r, s)$. By the intermediate value property, there exists an x between x_1 and x_2 such that $g(x, x_0) = y$. Thus $(x, x_0, g(x, x_0)) \in (a, b) \times \{x_0\} \times (r, s)$ and $h = g|(I \times \{x_0\})$ is dense in $I \times \{x_0\} \times I$. Clearly h is onto I.

Example 3: There is an almost continuous function $f : I \to I$ which has a perfect road and does not have the CIVP. Hence f is not extendable.

There exists a connectivity function $g : I^2 \to I$ such that for some $p \in I$ the restriction $h = g|(I \times \{p\})$ is dense in $I \times \{p\} \times I$ and onto I. If we let I be embedded in I^2 as $I \times \{p\}$, we can think of h as being an extendable function from $I \to I$. So h is almost continuous and has a perfect road. Also h has the SCIVP.

Let $a, b \in I$ with $h(a) \neq h(b)$, and let K be a Cantor set between $h(a)$ and $h(b)$. Since h has the SCIVP, the set $h^{-1}(K)$ contains a Cantor set. By transfinite induction there exists a set $B \subset h^{-1}(K)$ which contains no perfect set and meets every perfect subset of $h^{-1}(K)$ in c-many points and each point of B is a bilaterial limit point of B, see page 514 of [11].

A function f can be defined on B so that f meets every perfect set F in $I \times \{p\} \times I$ for which $\pi_1(F) \cap B$ contains c-many points, [4]. Let $f = h$ on $(I \times \{p\}) - B$.

We now show that f is almost continuous. Assume f is not. Then there exists a minimal blocking set H contained in $I \times \{p\} \times I$ such that the graph of f contains no point of H and $\pi_1(H)$ is connected [10]. Therefore $\pi_1(H) \cap B \neq \emptyset$. Otherwise, if $\pi_1(H) \cap B = \emptyset$, then the graph of h contains no point of H, a contradiction to h being almost continuous. Then $\pi_1(H) \cap B$ contains c-many points because each point of B is a bilateral limit point of B and $\pi_1(H)$ is connected. By construction, the graph of f meets H, a contradiction.

Let $x \in I \times \{p\}$. We now construct a perfect road at x for f. Choose closed intervals I_n in $I - K$ such that

$$\text{diameter}(I_n \cup \{f(x)\}) < 1/n.$$

For each n, select different points $y_n, y'_n \in I_n$. Since h is Darboux and its graph is dense in $I \times \{p\} \times I$, the graph of h meets the intervals

$$\begin{aligned}
&[x - 1/n, x] \times \{y_n\} && \text{in some point} && (x_n, y_n) = (x_n, h(x_n)),\\
&[x, x + 1/n] \times \{y_n\} && \text{in some point} && (z_n, y_n) = (z_n, h(z_n)),\\
&[x - 1/n, x] \times \{y'_n\} && \text{in some point} && (x'_n, y'_n) = (x'_n, h(x'_n)), \quad \text{and}\\
&[x, x + 1/n] \times \{y'_n\} && \text{in some point} && (z'_n, y'_n) = (z'_n, h(z'_n)).
\end{aligned}$$

Since $B \subset h^{-1}(K)$, x_n, z_n, x'_n, z'_n are not in B. Thus

$$\begin{aligned}
y_n &= h(x_n) = f(x_n),\\
y_n &= h(z_n) = f(z_n),\\
y'_n &= h(x'_n) = f(x'_n), \quad \text{and}\\
y'_n &= h(z'_n) = f(z'_n).
\end{aligned}$$

As $n \to \infty$, $x_n \to x$, $z_n \to x$, $x'_n \to x$, $z'_n \to x$, $f(x_n) \to f(x)$, $f(z_n) \to f(x)$, $f(x'_n) \to f(x)$, and $f(z'_n) \to f(x)$.

Since the extendable function h has the WCIVP, there exist two sequences of Cantor sets $C_1, C_2, \ldots$ and

$$C'_1, C'_2, \ldots \text{ in } (I \times \{p\}) - B$$

such that C_n is between x_n and x'_n, C'_n is between z_n and z'_n, $f(C_n) = h(C_n) \subset I_n$, and $f(C'_n) = h(C'_n) \subset I_n$. It follows that the set

$$P = (\bigcup_{n=1}^{\infty} C_n) \cup \{x\} \cup (\bigcup_{n=1}^{\infty} C_n') \text{ is a perfect set and}$$

$$f|P \text{ is continuous at } x.$$

Lastly, we show that f does not have the CIVP. Assume there exists a Cantor set C between a and b such that $f(C) \subset K$. Let $y \in I - K$. Since $C \times \{y\}$ is a perfect set and $C \cap B$ contains c-many points, f must meet $C \times \{y\}$. So there is some $t \in C$ such that $f(t) = y$ which implies that $f(C) \not\subset K$, a contradiction.

Question. Does there exist an almost continuous function $f : I \to I$ that has the SCIVP but is not extendable?

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