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Kurzweil–Henstock Integration and the Strong Lusin Condition

Lee Peng Yee [LPY] in connection with the study of ACG^* functions employed a condition which lies somewhere between absolute continuity and the Lusin condition N, and called it the strong Lusin condition. This condition was also studied by Kurzweil, Jarnik and Schwabik. The aim of this talk is to indicate how this condition can be used in an alternative approach to KH-integration. All functions in the sequel are real valued and measure always means Lebesgue measure.

Definition of SL. *A function F is said to satisfy the Strong Lusin Condition, or briefly SL, on a set S if for every set E of measure zero, and every positive ε there exists a $\gamma : S \to (0, \infty)$ such that for any γ-fine partial division $D = ([u,v], \xi)$ with $\xi \in E \cap S$ we have $(D) \sum |F(u,v)| < \varepsilon$, where $F(u,v)$ denotes $F(v) - F(u)$.*

An application of Vitali's covering theorem makes it possible to show that SL implies N. We denote by N_δ the set of zeros of a function δ. A function $\delta : [a,b] \to [0, \infty)$ will be called a gauge if N_δ is of measure zero.

Definition of the SL-integral. *A function f is said to be SL-integrable on $[a,b]$ if there exists an SL-function F and for every positive ε there is a gauge δ such that $|(D) \sum [f(\xi)(v-u) - F(u,v)]| < \varepsilon$ for every δ-fine partial division D of $[a,b]$. The number $F(a,b)$ is then the SL integral of f and it is denoted by $SL \int_a^b f$.*

Roughly speaking Henstock's lemma is already incorporated in the definition of the SL-integral, an idea already used by Pfeffer in his work on the Gauss-Green Theorem. It can be shown that F from the definition is uniquely determined (up to an additive constant) and that the SL-integral is well defined. It is easy to prove that a KH-integrable function is SL-integrable and it is possible to prove the converse. The KH and SL integrals are equivalent. The concept of the SL-integral allows some simplifications in some proofs of the KH theory. This perhaps could be seen from the theorems that follow. The proofs will appear elsewhere.

The Fundamental Theorem. *If $F' = f$ almost everywhere on $[a,b]$ and F is SL on $[a,b]$ then $\int_a^b f = F(b) - F(a)$.*

It is not too difficult to convince oneself that this theorem includes both Fundamental Theorems for KH-integral and the Lebesgue integral.

A family $\mathcal{F} = \{F_\lambda : \lambda \in \Omega\}$ is said to satisfy the strong Lusin condition uniformly on a set S, or briefly USL, if for every positive ε and every set of measure zero, $N \subset S$, there exists a $\gamma : S \to (0,\infty)$ such that (1) holds for every $F \in \mathcal{F}$ whenever $D = ([u,v],\xi)$ is a γ-fine partial division with $\xi \in N$.

A family $\mathcal{F} = \{f_\lambda : \lambda \in \Omega\}$ is said to be SL-equintegrable on [a,b] if there exist functions F_λ satisfying the SL condition and for every positive ε there is a gauge δ such that for all $\lambda \in \Omega$ $|\sum(f_\lambda(\xi)(v-u) - F_\lambda(u,v)| < \varepsilon$ whenever D is a δ-fine partial division of [a,b]. The basic convergence theorem for the SL integral is as follows :

Convergence Theorem. *If $\{f_n : n \in \mathrm{N}\}$ is SL-equintegrable on $[a,b]$, the family $\{F_n : n \in \mathrm{N}\}$ of SL-primitives is USL, and f_n converges almost everywhere to f, then $F_n(x) - F_n(a)$ converges to $F(x) - F(a)$ where F is a SL-primitive of f.*

The next result gives a connection with the equintegrability of Kurzweil [JK; p.40 Satz 5.2].

Equintegrability Theorem. *If $\{f_n : n \in \mathrm{N}\}$ is equintegrable on $[a,b]$, and the family $\{F_n : n \in \mathrm{N}\}$ of primitives is uniformly bounded on $[a,b]$, then $\{f_n : n \in \mathrm{N}\}$ is SL-equintegrable on $[a,b]$ and $\{F_n : n \in \mathrm{N}\}$ is USL. Conversely, if $\{f_n : n \in \mathrm{N}\}$ is SL-equintegrable on $[a,b]$, the family $\{F_n : n \in \mathrm{N}\}$ of primitives is USL, and $\{f_n(x) : n \in \mathrm{N}\}$ is bounded for each x, then $\{f_n : n \in \mathrm{N}\}$ is equintegrable on $[a,b]$.*

References

[LPY] Lee P.Y., On ACG* functions, Real Analysis Exchange 15(1989-90) 754-759.

[JK] J. Kurzweil, Nichtabsolut Konvergente Integrale, Teubner, Leipzig 1980 .