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The Integral Over Product Spaces, and Wiener's Formula

In the language of division space theory let T be a set of points t, $\mathcal{T}$ a suitable family of subsets of T, called *(generalized) intervals* I, and $\mathcal{U}^1$ a family of interval–point pairs (I,t), with $\mathbf{A}$ a suitable family of subsets $\mathcal{U}$ of $\mathcal{U}^1$. A function $h : \mathcal{U}^1 \mapsto \mathbf{C}$ (the *complex plane*) is called *ultimately finitely additive* if there is a $\mathcal{U} \in \mathbf{A}$ such that for the $(I,t) \in \mathcal{U}$, $h(I,t) = h(I)$, independent of t, and for every division from $\mathcal{U}$ of an interval J the sum of the $h(I)$ is $h(J)$. Now let h be $\mathbf{A}$-integrable over an elementary set E (a finite union of disjoint intervals). Thus, there is a number H and, given $\epsilon > 0$, there is a $\mathcal{U} \in \mathbf{A}$ such that for every division $\mathcal{E}$ of E from $\mathcal{U}$,

$$\left|(\mathcal{E})\sum h - H\right| < \epsilon,$$

where $(\mathcal{E})\sum$ denotes summation over $\mathcal{E}$. Then, assuming $\mathbf{A}$ suitable, h is $\mathbf{A}$-integrable, say to $H(I)$, over every interval I that can be used in a division of E (in particular, $I \subset E$) and for $\mathcal{E}$ from a small enough $\mathcal{U} \in \mathbf{A}$,

$$g(\mathcal{E}) \equiv (\mathcal{E})\sum |h - H| < \epsilon.$$

If h is ultimately finitely additive, $g(\mathcal{E}) = 0$ eventually. If h is <u>not</u> ultimately finitely additive, in each $\mathcal{U} \in \mathbf{A}$ there is a division $\mathcal{E}$ with $g(\mathcal{E}) > 0$, and this difference is used in a study of the integral over product spaces.

For $\mathcal{U} \subset \mathcal{U}^1$, $\mathrm{E}^*.\mathcal{U}$ is the set of all t with $(I,t) \in \mathcal{U}$ for some I that can be used in a division of E, and E^* is the intersection of $\mathrm{E}^*.\mathcal{U}$ for all $\mathcal{U} \in \mathbf{A}$ that contain a division of E. If T_u, $\mathcal{T}_u$, $\mathcal{U}_u^1$ are the constructions used in the integration processes for $u = x, y$, then for $u = z \equiv (x,y)$ we have $T_z = T_x \otimes T_y$, $\mathcal{T}_z$, is the family of $I_x \otimes I_y$ for all $I_u \in \mathcal{T}_u$ $(u = x, y)$, and $\mathcal{U}_z^1$ is the family of $(I_x \otimes I_y, (x,y))$, written $(I_x, x) \otimes (I_y, y)$, for all $(I_u, u) \in \mathcal{U}_u^1$ $(u = x, y)$. Then $\mathbf{A}_z$ is some family of subsets of $\mathcal{U}_z^1$ such that $\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z$ *have the Fubini property in common, i.e.* for E_u an arbitrary elementary set in $\mathrm{T}_u (u = x, y)$ with $\mathrm{E}_z = \mathrm{E}_x \otimes \mathrm{E}_y$ and arbitrary $\mathcal{U}_z \in \mathbf{A}$ containing a division of E_z, there is a $\mathcal{U}_y(.) : \mathrm{E}_x^* \mapsto \mathbf{A}_y$, and to each collection of divisions $\mathcal{E}_y(x)$

of E_y from $\mathcal{U}_y(x)$, one division for each such x, there is a $\mathcal{U}_x \in \mathbf{A}_x$ such that if $(I_x, x) \in \mathcal{U}_x$, $(I_y, y) \in \mathcal{E}_y(x)$, then $(I_x, x) \otimes (I_y, y) \in \mathcal{U}_z$. Also we assume the property obtained on interchanging x and y, leaving the product space as $\mathrm{T}_x \otimes \mathrm{T}_y$.

Theorem 1 *Suppose that $\mathbf{A}_x, \mathbf{A}_y, \mathbf{A}_z$ have the Fubini property in common, with $(T_u, \mathcal{T}_u, \mathbf{A}_u)$ $(u = x, y)$ fully decomposable. Let E_u $(u = x, y)$ be elementary sets with $E_z = E_x \otimes E_y$.*

1. *Let $h_x(I_x, x)h_y(x; I_y, y)$ be $\mathbf{A}_z$-integrable over E_z and let X be the set of x for which $h_y(x; .)$ is ultimately finitely additive. Then h_x is VBG* in $E_x^* \backslash X$.*

2. *Let $h_y(I_y, y)h_x(y; I_x, x)$ be $\mathbf{A}_z$-integrable over E_z and let Y be the set of y for which $h_x(y; .)$ is ultimately finitely additive. Then h_y is VBG* in $E_y^* \backslash Y$.*

The usual product $f(x, h)h_x(I_x, x)h_y(I_y, y)$ can be used in 1. and 2., and conversely. The converse of *Theorem 1* encounters a Sierpiǹski construction, a plane non-measurable set that meets every line parallel to the x and y axes in at most two points. So the converse can only be partial, with no easy proof except in simple cases.

Theorem 2 *Let $(T_z, \mathcal{T}_z, \mathbf{A}_z)$ be the product division space of the fully decomposable additive division spaces $(T_u, \mathcal{T}_u, \mathbf{A}_u)$ $(u = x, y)$. For every i nterval $I_x \otimes I_y$ in divisions of $E_z = E_x \otimes E_y$ let there be elementary sets E_{1u} disjoint from I_u with $I_u \cup E_{1u} = E_u (u = x, y)$. Let h_u be $\mathbf{A}_u$-integrable to $H_u(I_u)$ over I_u and let M_u be an arbitrarily small finitely additive majorant of $|h_u - H_u|$ $(u = x, y)$. Then $h_x h_y$ is $\mathbf{A}_z$-integrable to $H_x(E_x)H_y(E_y)$ over E_z in the following cases:*

1. *if h_u is ultimately finitely additive over E_u, VBG^* or not $(u = x, y)$;*

2. *if h_u is VBG^* over E_u^* when h_u is, but h_v is not, ultimately finitely additive over its elementary set $(u = x$ and $v = y$ and $v = x)$;*

3. *if h_u is not ultimately finitely additive over E_u but is VBG^* over $E_u^*(u = x, y)$.*

Some of these results generalize Wiener's formula for Lebesgue integration with Wiener measure over infinite dimensional Cartesian product spaces of copies of the real line, and are useful in Feynman integration.