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THE RADON-NIKODYM DERIVATIVE IN EUCLIDEAN SPACES

The usual Radon-Nikodym Theorem can be stated as follows: If Φ is a signed measure and m is a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X where both Φ and m are σ-finite, then there is an m-measurable function f and a singular completely additive function of a set Θ such that for each $E \in \mathcal{A}$, $\Phi(E) = \int_E f\,dm + \Theta(E)$. Here Θ is singular means that there is a set $Z \in \mathcal{A}$ with $m(Z) = 0$ such that for $B \in \mathcal{A}$ with $B \cap Z = \phi$, $\Theta(B) = 0$. The function f is sometimes called the Radon-Nikodym derivative of Φ with respect to m and written $d\Phi/dm$.

In Euclidean spaces, if a function Φ is defined on the Borel sets, the general upper derivate is defined by $\bar{D}\Phi(x) = \sup \lim \Phi(E_n)/m(E_n)$ where the supremum is taken over all regular sequences $\{E_n\}$ of closed sets with $x \in \bigcap E_n$ and $\lim \operatorname{diam} E_n = 0$ for which $\lim \Phi(E_n)/m(E_n)$ exists. A sequence $\{E_n\}$ is regular provided there is $r > o$ such that for each n $m(E_n)/m(Q_n) > r$ where Q_n is the smallest cube containing E_n. The general lower derivate $\underline{D}\Phi$ is defined by the infimum of such limits and the general derivative $D\Phi(x)$ is the common value of $\bar{D}\Phi(x)$ and $\underline{D}\Phi(x)$ when they are equal.

Saks ([1] p. 10 ff.) considers only countably additive set functions Φ satisfying the restriction that Φ is finite valued. None of the results in Saks are lost if these functions Φ are considered to be countably additive and finite valued on bounded sets. In the general theory of countably additive set functions, the Jordan decomposition theorem asserts that any completely additive set function Φ can be written as $\Phi^+ - \Phi^-$ where Φ^+ and Φ^- are non-negative. Moreover, only one of Φ^+ or Φ^- can take on the value $+\infty$. Thus it will only be necessary to consider non-negative functions Φ. We also consider below only the case where the σ-algebra $\mathfrak{a}$ is the collection of Borel sets.

Under the restriction that Φ be finite valued on bounded sets, one obtains the following results ([1] p.69 and p.114 ff.):

1) Given $E \in \mathfrak{a}$ and $\varepsilon > o$, there is a closed set $F \subset E$ such that $\Phi(E \setminus F) < \varepsilon$
2) The general derivative exists a.e. and equals $d\Phi/dm$ a.e.
3) If Θ is singular, $D\Theta = 0$ a.e.

We wish to consider the consequences in Euclidean spaces of omitting the hypothesis that Φ be finite valued on bounded sets. (The function Φ will be assumed to be σ-finite, non-negative and defined on the Borel sets.) Easy examples 1, 2, and 3 given below show that each of 1), 2), and 3) above respectively fails to hold. However, it will also be shown that weaker versions of 1), 2), and 3) do hold and these indicate what happens when Φ is not finite valued on bounded sets.

Example 1. Let E be the set of points (x, y) in the plane with $y = \sin(1/x)$. Let $\Phi(A)$ be the length of $A \cap E$. Then given any $\varepsilon > 0$ there does not exist a closed set $F \subset E$ with $\Phi(E \setminus F) < \varepsilon$. This is because any set $A \subset E$ with $\Phi(E \setminus A) < \varepsilon$ must have points on the y-axis in its closure.

Example 2. Let P be a nowhere dense perfect subset of $[0, 1]$ of positive measure. Let (a_n, b_n) be the intervals contiguous to P in $[0, 1]$. Let $f(x) = 1/(b_n - a_n)$ if $x \in (a_n, b_n)$ and $f(x) = 0$ if $x \in P$. Let

$\Phi(E) = \int_E f(x)dm$. Then $f = d\Phi/dm$ but $\bar{D}\Phi(x) = \infty$ at each $x \in P$ because every closed interval with x in its interior contains infinitely many contiguous intervals and hence has infinite Φ-measure due to the fact that $\Phi((a_n, b_n)) = 1$.

Example 3. Let $\{E_n\}$ be a sequence of pairwise disjoint sets of measure 0 with $\lim \operatorname{diam} E_n = 0$ and such that for each natural number k, $\bigcup_{n=k}^{\infty} E_n$ is dense. Let Φ be a measure such that for each n, $\Phi(E_n) = 1$ and $\Phi((\bigcup E_n)^c) = 0$. Then Φ is singular, but $D\Phi(x) = \infty$ at every point x since every neighborhood of each point x has infinite Φ measure.

Clearly, variations on these examples are possible. However, given Φ which is σ-finite and defined on the Borel sets, let $A_\Phi = \{x$: every neighborhood of x has infinite Φ measure$\}$. Then A_Φ is a closed set. Moreover, every Borel set can be approximated within ε by its closed subsets if and only if $A_\Phi = \phi$. For if $A_\Phi \neq \phi$ then for any $x \in A_\Phi$, $\{x\}$ is of finite Φ-measure because Φ is σ-finite. Then $\{x\}^c$ can not be approximated with a closed subset of $\{x\}^c$. Indeed, every set $E \subset \{x\}^c$ with $\Phi(E^c) < \infty$ has x in its closure. Thus the restriction of Φ to functions which are finite valued on bounded sets is both necessary and sufficient for 1) to hold. (Conversely, of course, no open set G with $x \in G$ approximates $\{x\}$ in measure.)

To characterize those sets E for which $\Phi(E)$ can be approximated within ε using closed subsets of E, let $A_{\Phi,E} = \{x$: every neighborhood N of x satisfies $\Phi(N \cap E) = \infty\}$.

1') Given E and any $\varepsilon > o$ there is a closed subset F of E with $\Phi(E \setminus F) < \varepsilon$ if and only if $A_{\Phi,E} \subset E$.

For if $A_{\Phi,E} \setminus E \neq \phi$, the same argument as above shows that E can not be approximated from within by closed sets. Suppose $A_{\Phi,E} \subset E$. For each pair of natural numbers n and k, let $A_{n,k} = \{x \in E: 1/k \leq d(x, A_{\Phi,E}) < 1/(k-1)$ and $n-1 \leq ||x|| < n\}$ where $1/0$ is understood to be ∞. Then each $A_{n,k}$ is of finite measure, can be approximated from within by closed sets $F_{n,k}$ so that $\Phi(A_{n,k} \setminus F_{n,k}) < \varepsilon/2^{n+k+2}$ and then $F = \bigcup F_{n,k} \cup A_{\Phi,E}$ is closed and $\phi(E \setminus F) < \varepsilon$. The set F is closed because $A_{\Phi,E}$ is closed and $\bigcup F_{n,k}$ can not have any limit points other than points in $A_{\Phi,E}$.

Now if Φ takes on the value $+\infty$ on some bounded set, even though 2) and 3) need not hold,

2') $\underline{D}\Phi = d\Phi/dm$ a.e.

and if θ is a singular function which takes on the value $+\infty$

3') $\underline{D}\Theta = 0$ a.e.

(Conversely, if Φ takes on $-\infty$, $\bar{D}\Phi = d\Phi/dm$ a.e. and if Θ is singular and takes on $-\infty$, $\bar{D}\Theta = 0$ a.e.)

Note that the negative part of a completely additive set function which takes on $+\infty$ is bounded below and hence its general derivative exists and is finite a.e. and equals its Radon-Nikodym derivative a.e.; in the case where the function is singular, both derivatives equal 0 a.e.

To see that 3') holds, note that there is a set Z of measure 0 such that for each $E \subset Z^c$, $\Theta(E) = 0$. But each point $x \in Z^c$ is a point of density of Z^c and thus there is a closed set $E_x \subset Z^c$ so that x is a point of density of E_x. Then if I_n is the sequence of intervals centered at x with diameter $1/n$, $E_x \cap I_n$ satisfies $\lim m(E_x \cap I_n)/m(I_n) = 1$. It follows that $\underline{D}\Theta(x) = 0$ because $\Theta(E_x \cap I_n)/m(E_x \cap I_n) = 0$ and $\{E_x \cap I_n\}$

is a regular sequence of closed sets.

To see that 2') holds, let Φ be given and let $f = d\Phi/dm$. Then

$\Phi(E) = \int_E fdm + \Theta(E)$ and since $\underline{D}\Theta(x) = 0$ a.e., it suffices

to show that $\Phi_o(E) = \int_E fdm$ satisfies $\underline{D}\Phi_o = f$ a.e. To see this, for each natural number n, let $f_n(x) = \min(f(x),n)$. Let $\Phi_n(E) = \int_E f_n(x)dm$. Then $\{f_n(x)\}$ is a non-decreasing sequence of functions which approaches $f(x)$. For each Borel set E, the monotone convergence theorem implies that $\lim \Phi_n(E) = \Phi_o(E)$. Moreover, since each Φ_n is finite valued on bounded sets, $D\Phi_n$ exists a.e. Clearly, $\underline{D}\Phi_o(x) \geq D\Phi_n(x)$. For if x belongs to each closed set E_k where the sequence $\{E_k\}$ is regular and $\lim \operatorname{diam} E_k = 0$, then $\Phi_o(E_k)/m(E_k) \geq \Phi_n(E_k)/m(E_k)$. Since Φ_o is a σ-finite measure, it follows that f is finite a.e. For otherwise, if f equaled ∞ on a set of positive measure E, Φ would be infinite on each set of positive measure of E and 0 on each subset of measure 0 of E and then E would not be the countable union of sets of finite Φ-measure. Finally, to see that $\underline{D}\Phi(x) = f(x)$ a.e., let $A_n = \{x: f_n(x) = f(x) < n$ and $D\Phi_n(x) = f_n(x)\}$. Let x be a point of density of A_n. Then there is a closed set E_x with density 1 at x such that $E_x \subset A_n$. If I_k is the closed interval with equal length sides of diameter $1/k$ centered at x and $E_k = E_x \cap I_k$, then E_k is a regular sequence of closed sets and

$$\underline{D}\Phi(x) \leq \underline{\lim} \int_{E_k} f \, dm/m(E_k) = \lim \int_{E_k} f_n dm/m(E_k)$$
$$= f_n(x) = f(x).$$

Since almost every point x is a point of density of some A_n, it follows that $\underline{D}\Phi(x) \leq f(x)$ a.e. and hence $\underline{D}\Phi(x) = f(x)$ a.e.

The following simple corollary is worth noting: if $m(A\Phi) = 0$ and $\Phi(E) = \int f dm + \Theta(E)$ with Θ singular, then $d\Phi/dm = D\Phi$ a.e. and $D\Phi = 0$ a.e. Indeed, since $A\Phi$ is closed, it will be avoided in the computation of either $D\Phi(x)$ or $D\Theta(x)$ whenever $x \in A\Phi$. Thus, on $A\Phi^c$, $D\Phi = d\Phi/dm$ a.e. and $D\Phi = 0$ a.e.

REFERENCES:

1. S. Saks, Theory of the Integral, 2nd ed. revised, New York.

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