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ANOTHER PROOF OF THE MEASURABILITY OF δ FOR THE GENERALIZED RIEMANN INTEGRAL

The purpose of this paper is to show that restricting the function δ in the generalized Riemann integral to be measurable does not change the nature of the integral. The two definitions that follow will clarify the problem.

DEFINITION 1: Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A tagged interval $(s,[c,d])$ consists of an interval $[c,d]$ in $[a,b]$ and a point s in $[c,d]$. The tagged interval $(s,[c,d])$ is subordinate to δ if $[c,d] \subset (s-\delta(s), s+\delta(s))$. Let $\mathcal{P} = \{(s_i,[c_i,d_i]) : 1 \leq i \leq N\}$ be a finite collection of non-overlapping tagged intervals in $[a,b]$. If $(s_i,[c_i,d_i])$ is subordinate to δ for each i, then we write $\mathcal{P}$ is subordinate to δ. If in addition $\mathcal{P}$ is a partition of $[a,b]$, then we write $\mathcal{P}$ is subordinate to δ on $[a,b]$. For a function $f:[a,b] \to R$ and a function F defined on the intervals of $[a,b]$, we write

$$f(\mathcal{P}) = \textstyle\sum_i f(s_i)(d_i - c_i) \quad \text{and} \quad F(\mathcal{P}) = \sum_i F([c_i,d_i]).$$

DEFINITION 2: The function $f:[a,b] \to R$ is GR (mGR) integrable on $[a,b]$ if there exists a real number α with the following property: for each $\epsilon > 0$ there exists a positive (positive, measurable) function δ on $[a,b]$ such that $|f(\mathcal{P}) - \alpha| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ on $[a,b]$. The function f is GR (mGR) integrable on the set $E \subset [a,b]$ if $f\chi_E$ is GR (mGR) integrable on $[a,b]$.

It is clear that every mGR integrable function is GR integrable and that the integrals are equal. We will show that every GR integrable function is mGR integrable. We first establish some notation. Given a point t and a set E, CE is the complement of E, $\mu(E)$ is the Lebesgue measure of E, χ_E is the characteristic function of E, and $\rho(t,E)$ is the distance from t to E. We will use $\omega(f,I)$ to denote the oscillation of the function f on the interval I.

The mGR integral shares many of the properties of the GR integral, including integrability on subintervals and Henstock's Lemma. By easy adaptations of the proofs for the GR integral, we obtain the next two results.

THEOREM 3: If $f:[a,b] \to R$ is mGR integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is mGR integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

THEOREM 4: Suppose that $f:[a,b] \to R$ is mGR integrable on each interval $[\alpha,\beta] \subset (a,b)$. If $\int_\alpha^\beta f$ converges to a finite limit as $\alpha \to a^+$ and $\beta \to b^-$, then f is mGR integrable on $[a,b]$ and $\int_a^b f = \lim_{\substack{\alpha \to a^+ \\ \beta \to b^-}} \int_\alpha^\beta f$.

THEOREM 5: If $f : [a,b] \to R$ is Lebesgue integrable on $[a,b]$, then f is mGR integrable on $[a,b]$ and the integrals are equal.

PROOF: Let $\epsilon > 0$ and choose a positive number $\eta < \epsilon/3$ such that $\int_A |f| < \epsilon/3$ whenever $\mu(A) < \eta$. Let $\beta = \min\{1, \epsilon/3(\eta + b - a)\}$. Now $[a,b] = \cup_n E_n$ where

$$E_n = \{t \in [a,b] : (n-1)\beta < f(t) \leq n\beta\}$$

for each integer n. Note that each E_n is measurable and that the E_n's are disjoint. For each n, choose an open set G_n such that $E_n \subset G_n$ and

$$\mu(G_n - E_n) < \eta/\left(2^{|n|}3(|n|+1)\right).$$

Define δ on $[a,b]$ by $\delta(t) = \rho(t, CG_n)$ for $t \in E_n$. Then δ is a positive, measurable function and $t \in E_n$ implies $(t - \delta(t), t + \delta(t)) \subset G_n$. Proceeding as in the proof by Davies and Schuss [1], we find that $|f(\mathcal{P}) - (L)\int_a^b f| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ on $[a,b]$. Hence, the function f is mGR integrable on $[a,b]$ and $\int_a^b f = (L)\int_a^b f$.

The proof of the next theorem for the GR integral is not so well-known. We include the details for completeness.

THEOREM 6: Let E be a bounded, closed set with bounds a and b and let $\{(a_k, b_k)\}$ be the sequence of intervals contiguous to E in $[a,b]$. Suppose that $f : [a,b] \to R$ is mGR integrable on E and on each interval $[a_k, b_k]$. If the series $\sum_k \omega(\int_{a_k}^t f, [a_k, b_k])$ has a finite sum, then f is mGR integrable on $[a,b]$ and $\int_a^b f = \int_a^b f\chi_E + \sum_k \int_{a_k}^{b_k} f$.

PROOF: Since the function $f\chi_E$ is mGR integrable on $[a,b]$, it is sufficient to prove that the function $g = f - f\chi_E$ is mGR integrable on $[a,b]$ and that $\int_a^b g = \sum_k \int_{a_k}^{b_k} f$. For each t in $[a,b]$, let $I_t = [a,t]$ and define a function $G : [a,b] \to R$ by

$$G(t) = \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f\chi_{I_t}.$$

The series converges uniformly by the Weierstrass M-test and each of the functions $\int_{a_k}^{b_k} f\chi_{I_t}$ is continuous on $[a,b]$. Therefore, the function G is continuous on $[a,b]$. We will treat G as a finitely additive function defined on the subintervals of $[a,b]$, that is, $G([c,d]) = G(d) - G(c)$. Note that $G([a,b]) = \sum_k \int_{a_k}^{b_k} f$ and that $\int_{a_k}^{b_k} g = \int_{a_k}^{b_k} f = G([a_k, b_k])$ for each k.

Let $\epsilon > 0$. For each k, choose a positive, measurable function δ_k on $[a_k, b_k]$ so that $|g(\mathcal{P}) - G([a_k, b_k])| < \epsilon 2^{-k-2}$ whenever $\mathcal{P}$ is sub δ_k on $[a_k, b_k]$. Choose a positive integer N such that $\sum_N^\infty \omega(\int_{a_k}^t f, [a_k, b_k]) < \epsilon/4$. Let $I_k = (a_k, b_k)$ and let $E_0 = \bigcup_1^{N-1}\{a_k, b_k\}$. Since G is continuous, there exists a positive function δ_0 on E_0 such that $|G(\mathcal{P})| < \epsilon/4$ whenever $\mathcal{P}$ is sub δ_0 and all of the tags of $\mathcal{P}$ are in E_0. Define a positive function δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \min\{\delta_k(t), \rho(t, CI_k)\}, & \text{if } t \in (a_k, b_k); \\ \rho(t, E_0), & \text{if } t \in E - E_0; \\ \delta_0(t), & \text{if } t \in E_0; \end{cases}$$

and note that δ is measurable. Now suppose that $\mathcal{P}$ is subordinate to δ on $[a,b]$ and assume that all of the tags are endpoints. Let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ that has tags in E_0, let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in $E - E_0$, and for each k, let $\mathcal{P}_k$ be the subset of $\mathcal{P} - (\mathcal{P}_0 \cup \mathcal{P}_E)$ that has intervals in $[a_k, b_k]$. Each $\mathcal{P}_k$ is sub δ_k and $\mathcal{P}_0$ is sub δ_0. Furthermore $I \cap (a_k, b_k) = \emptyset$ for $1 \le k < N$ for each interval I in $\mathcal{P}_E$. Since the tags of $\mathcal{P}_E$ are in E, for each $k \ge N$, the interval (a_k, b_k) intersects at most two intervals in $\mathcal{P}_E$. Let $\pi = \{k : \mathcal{P}_k \ne \emptyset\}$ and use Henstock's Lemma to compute

$$\begin{aligned}
|g(\mathcal{P}) - G([a,b])| &= \left| g(\mathcal{P}_E) + g(\mathcal{P}_0) + \sum_{\pi} g(\mathcal{P}_k) - G(\mathcal{P}_E) - G(\mathcal{P}_0) - \sum_{\pi} G(\mathcal{P}_k) \right| \\
&\le \sum_{\pi} |g(\mathcal{P}_k) - G(\mathcal{P}_k)| + |G(\mathcal{P}_E)| + |G(\mathcal{P}_0)| \\
&\le \sum_{\pi} \epsilon\, 2^{-k-2} + 2 \sum_{k=N}^{\infty} \omega\left(\textstyle\int_{a_k}^{t} f, [a_k, b_k]\right) + \epsilon/4 \\
&< \epsilon/4 + 2\epsilon/4 + \epsilon/4 \\
&= \epsilon.
\end{aligned}$$

Therefore, the function g is mGR integrable on $[a,b]$ and $\int_a^b g = \sum_k \int_{a_k}^{b_k} f$. This completes the proof.

We need two other results. The first is an easy consequence of the fact that the indefinite GR integral is ACG_*. For a proof, see Saks [5]. The second is a straight-forward application of the Heine-Borel Theorem. A proof can be found in Romanovski [4].

THEOREM 7: Let $f : [a,b] \to R$ be GR integrable on $[a,b]$ and let E be a perfect set in $[a,b]$. Then there exists an interval $[c,d]$ with $c, d \in E$ and $E \cap (c,d) \ne \emptyset$ such that f is Lebesgue integrable on $E \cap [c,d]$. In addition, letting $[c,d] - E = \cup_n (c_n, d_n)$, we have

$$\sum_n \omega\left(\textstyle\int_{c_n}^{t} f, [c_n, d_n]\right) < \infty \qquad \text{and} \qquad \textstyle\int_c^d f = \int_c^d f\chi_E + \sum_n \int_{c_n}^{d_n} f.$$

LEMMA 8: (Romanovski's Lemma) Let $\mathcal{F}$ be a family of open intervals in (a,b) and suppose that $\mathcal{F}$ has the following properties:
(1) If (α,β) and (β,γ) belong to $\mathcal{F}$, then (α,γ) belongs to $\mathcal{F}$.
(2) If (α,β) belongs to $\mathcal{F}$, then every open interval in (α,β) belongs to $\mathcal{F}$.
(3) If (α,β) belongs to $\mathcal{F}$ for every interval $[\alpha,\beta] \subset (c,d)$, then (c,d) belongs to $\mathcal{F}$.
(4) If all of the intervals contiguous to the perfect set $E \subset [a,b]$ belong to $\mathcal{F}$, then there exists an interval I in $\mathcal{F}$ such that $I \cap E \ne \emptyset$.
Then $\mathcal{F}$ contains the interval (a,b).

THEOREM 9: If $f : [a,b] \to R$ is GR integrable on $[a,b]$, then f is mGR integrable on $[a,b]$ and the integrals are equal.

PROOF: For each t in $[a,b]$, let $F(t) = (GR)\int_a^t f$. Let $\mathcal{F}$ be the collection of all open intervals (c,d) in (a,b) such that f is mGR integrable on $[c,d]$ and $\int_c^t f = F(t) - F(c)$ for all t in $[c,d]$. We must show that (a,b) belongs to $\mathcal{F}$. It is sufficient to prove that $\mathcal{F}$ satisfies the four conditions of Romanovski's Lemma. It is clear that $\mathcal{F}$ satisfies condition (2). Condition (1) follows from Theorem 3. Suppose that (α,β) belongs to $\mathcal{F}$ for each interval $[\alpha,\beta] \subset (c,d)$. Since $\int_\alpha^\beta f = (GR)\int_\alpha^\beta f$ converges to $F(d) - F(c)$ as $\alpha \to c^+$ and $\beta \to d^-$, the function f is mGR integrable on $[c,d]$ and $\int_c^d f = F(d) - F(c)$ by Theorem 4. It follows easily that (c,d) belongs to $\mathcal{F}$. Hence, condition (3) is satisfied.

Now let E be a perfect set in $[a,b]$ such that each interval contiguous to E in $[a,b]$ belongs to $\mathcal{F}$. By Theorem 7, there exists an interval $[c,d]$ with $c,d \in E$ and $(c,d) \cap E \neq \emptyset$ such that f is Lebesgue integrable on $E \cap [c,d]$ and the series $\sum_n \omega(\int_{c_n}^t f, [c_n,d_n])$ converges, where $[c,d] - E = \cup_n (c_n,d_n)$. In view of Theorem 5, we see that the hypotheses of Theorem 6 are satisfied. Hence, the function f is mGR integrable on $[c,d]$ and

$$\int_c^d f = (L)\int_c^d f\chi_E + \sum_n \int_{c_n}^{d_n} f = (GR)\int_c^d f = F(d) - F(c).$$

Similar reasoning is valid for the subintervals of (c,d) and it follows that (c,d) belongs to $\mathcal{F}$. Hence, condition (4) is satisfied and this completes the proof.

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