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SOME SYMMETRIC COVERING LEMMAS

The first significant investigation of the symmetric derivative was that of Khintchine [9] in 1927. As well as obtaining some elementary properties by elementary and unoriginal means he introduces the first interesting and new technique into the study in order to obtain the fact that a measurable function with a symmetric derivative on a set is differentiable almost everywhere on that set. Basically his argument is that if there is a uniform estimate on the symmetric difference quotient of a function f on a set E then at points that are both density points of E and points of approximate continuity of f a similar estimate for the ordinary difference quotients is available. This kind of an argument has been repeated often in subsequent years. For example in Stein and Zygmund [16, p. 266] this method is used to prove that measurable functions are continuous at almost every point of measurable sets on which they are symmetrically continuous; while they do not acknowledge Khintchine specifically as a source of the techniques they use, the ideas are easily traced to him and they suggest that the result had been known for some time.

A basic ingredient of the Khintchine proof is the use of points of approximate continuity and so the technique applies primarily to measurable functions. Thus, for example, something new is needed in order to investigate the points of symmetric continuity of a function that is not given *a priori* to be measurable. For a function f that is *everywhere* symmetrically continuous classical methods can still be made to work. Let $\omega_f(x)$ be the oscillation of f at x. Then ω_f is symmetrically continuous too and it is measurable. Thus the Stein-Zygmund theorem shows that it must be a.e. continuous. Together with a theorem of Fried [8] this shows that f is a.e. continuous too. This argument is due to David Preiss and can be considered to replace that in [11].

More recently, no doubt motivated by the arguments in [9] and [11], Uher [20] with some considerable technical skill has refined these techniques further still in order to give apparently definitive answers to questions of this nature in the study of symmetric continuity and symmetric derivatives. In this article we wish to introduce (or re-introduce) the reader to these arguments by presenting them in the form of covering lemmas, thus continuing the program suggested in [17] and [13]. Among the applications considered are some well known theorems as well as a few that seem to be new. The proofs are given in separate sections later in the article.

1 Covering relations

We have elsewhere in this Exchange ([17], [13]) defined a *symmetric full cover* on the real line to mean a collection $\mathcal{S}$ of closed intervals with the property that for every real x there is a $\delta(x) > 0$ so that

$$[x-h, x+h] \in \mathcal{S}$$

for every $0 < h < \delta(x)$. Such a notion evidently arises naturally in a study of symmetric limits. For example if a function f is everywhere symmetrically continuous then the collection of intervals

$$\{[a,b] : |f(b) - f(a)| < \epsilon\}$$

is a symmetric full cover of the real line for every $\epsilon > 0$.

There are some technical problems in adopting this definition that are resolved by a slight shift of viewpoint. It is traditional in derivation theory to convert notions of interval functions by considering an interval function $[a,b] \to h([a,b])$ as instead a function of two variables $h(a,b)$ defined in the plane $\mathbf{R}^2$. Then the interval $[a,b]$ corresponds to the planar point (a,b) and limit processes on the line correspond to limit process in the plane. If we wish to focus only on symmetric type limit processes it is often more natural to associate the interval $[a,b]$ more closely with its midpoint by selecting the point $((a+b)/2, (b-a)/2)$ to represent this interval. Thus we think of the scheme

$$(x,h) \longleftrightarrow [x-h, x+h]$$

or

$$[a,b] \longleftrightarrow ((a+b)/2, (b-a)/2).$$

The point (x, h) is the vertex of an isoceles right-angled triangle whose hypotenuse is the interval $[x - h, x + h]$ on the x–axis. We say (x, h) is the *vertex* for the interval $[x - h, x + h]$ and that the interval $[x - h, x + h]$ and the vertex (x, h) are *associated.*

Thus we convert from collections of intervals to subsets of $\mathbf{R}^2$ but retain the notion of coverings. Occasionally we refer to subsets of $\mathbf{R}^2$ as *vertex sets.* We shall allow vertex sets to contain points below the x–axis; the point (x, h) with $h < 0$ can still play a role in some of our definitions and will usually indicate that the interval $[x + h, x - h]$ is traversed in the reverse direction in some sense.

Definition 1 A set $V \subset \mathbf{R}^2$ is said to be a *full symmetric cover* for a set $E \subset \mathbf{R}$ provided that for every $x \in E$ there is a positive number $\delta(x)$ so that

$$0 < t < \delta(x) \Longrightarrow (x, t) \in V.$$

In order for a convenient expression of our covering theorems we require a notation for a theorem like that in [13] in terms of the vertex set. If we wish to have partitions such as

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

we can adopt the following scheme.

Definition 2 Let $V \subset \mathbf{R}^2$ be a vertex set and a and b real numbers. We write

$$V : a \rightsquigarrow b$$

if there is a finite sequence of numbers $\{x_0, x_1, \ldots, x_{n-1}, x_n\}$ so that $a = x_0$, $b = x_n$ and each pair

$$((x_{i+1} + x_i)/2, (x_{i+1} - x_i)/2) \in V.$$

If the sequence may be chosen increasing (i.e. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$) then we say that V allows a partition of the interval $[a, b]$. Note that in the definition each of the transitions $x_i \rightsquigarrow x_{i+1}$ is obtained by a reflection about the midpoint, above which the vertex point must belong to V. If $x_i > x_{i+1}$ then this reflection goes backwards and requires, in this definition, that the vertex point in V lie in the lower half plane below the midpoint.

2 Density covering lemmas

The classical theorem of Khintchine [9, p. 217] was the first in a series of studies devoted to investigating the fundamental properties of measurable functions on which a symmetry condition is imposed; it was proved there that a measurable function that is symmetrically differentiable on a set must be almost everywhere in that set differentiable in the ordinary sense. The geometrical arguments involved in the proof of that theorem may be (over-simply) described as noting that a symmetric covering relation that holds uniformly on some set E imposes strong conditions at the density points of E. Similar ideas have reappeared in later works, for example in [16], [2, Lemma 1, p. 17] and [20, Lemma 2, pp. 426–427].

We can present the essence of these arguments as a single covering lemma involving the density points of a set on which some uniform covering relation is given; because the idea is implicit in the article of Khintchine [9] we shall attribute the lemma to him.

We follow Uher [20] in presenting this in both a measure density and a category density version using the same arguments. For the measure-theoretic version the notation $d(E)$ denotes the set of (exterior) density points of E; that is to say

$$d(E) = \left\{ x : \lim_{h \to 0} \frac{|E \cap (x-h, x+h)|}{2h} = 1 \right\}.$$

In the category version we take, as in [20],

$$c(E) = \mathbf{R} \setminus \text{Closure}\left(\bigcup \{(a,b) : (a,b) \cap E \text{ is first category}\} \right).$$

(Note that this set is somewhat smaller than the analogous notion in [10, p. 83] being, in the language of that treatise, the interior of the set of points of second category.) In the statements of the lemmas and in their proofs we write $b(E)$ to denote either $d(E)$ the set of density points of E or, alternatively, $c(E)$ the set of category density points of E. With either interpretation we have the following covering lemma that it is appropriate to attribute to Khintchine.

Lemma 3 (Khintchine) *Let $E \subset \mathbf{R}$, $V \subset \mathbf{R}^2$ and $\eta > 0$. We suppose that*

$$x \in E,\ \ 0 < t < \eta \Longrightarrow (x,t) \in V$$

and that $z \in b(E)$. *Then there is a positive number* δ *so that for every point* $0 < |z - x| < \delta$ *there is a set* $A = A(x)$ *with* $z \in b(A)$ *such that for all* $a \in A$

$$z < a < x \Longrightarrow V : a \rightsquigarrow x$$

and

$$z > a > x \Longrightarrow V : x \rightsquigarrow a$$

by two reflections in V.

The covering argument of Lemma 3 has been used in some form frequently in the literature. For the most part it is limited in usefulness to obtaining the properties of measurable functions. In order to remove the measurability assumption far more subtle geometrical arguments are needed. The most penetrating analysis has been provided by Uher in his articles [19] and [20]. We present here the main covering argument of Uher (from [20, Lemma 1, pp. 423–426]) in the form of a covering lemma which has a number of important applications. As before for any set E let $b(E)$ denote either $d(E)$ the set of density points of E or, alternatively, $c(E)$ the set of "category density " points of E. In the statement of the theorem if $b(E) = c(E)$ then the phrase for "for almost every point $x \in b(E)$" must be interpreted in the category sense, i.e. as "for all but a first category subset of $c(E)$". Under either of these interpretations we have the following theorem which is directly attributable to Uher.

Lemma 4 (Uher) *Let* $E \subset \mathbf{R}$, $V_1, V_2 \subset \mathbf{R}^2$ *and* $\eta > 0$. *We suppose that* $V = V_1 \cup V_2$, *that*

$$x \in E,\ \ 0 < t < \eta \Longrightarrow (x,t) \in V_1$$

and for almost every point $x \in b(E)$ *there is a positive number* $\delta(x)$ *so that either*

$$0 < t < \delta(x) \Longrightarrow (x,t) \in V_2$$

or alternatively

$$0 > t > -\delta(x) \Longrightarrow (x,t) \in V_2.$$

Then for any $z \in b(E)$ *there is a neighborhood* U *of* z *(which depends only on* z, E *and* η*) so that for any* $x \in U$

$$z < x \Longrightarrow V : z \rightsquigarrow x$$

and

$$x < z \Longrightarrow V : x \rightsquigarrow z$$

by three reflections in V_1 *and two in* V_2.

3 Applications

3.1 Some preliminaries

We shall require a number of weak continuity conditions in our study. The following definition is a version of two forms of continuity known in the literature as qualitative-continuity and quasi-continuity. (For these notions, due to S. Marcus and S. Kempisty respectively, see [18, p. 21 and p. 26].) Remember that $c(A)$ denotes the set of category density points defined in the preceding section.

Definition 5 A function f is said to be *quasi-qualitatively continuous* (or simply qq–continuous) at a point x if

$$\{y : |f(x) - f(y)| < \epsilon\} \cap A \neq \emptyset$$

for every positive number ϵ and for every set A with $x \in c(A)$.

In order to better understand the notion recall that in order for f to be continuous at a point x then for every $\epsilon > 0$ the set

$$\{y : |f(x) - f(y)| < \epsilon\}$$

contains an open interval about x. In order for f to be quasi-continuous there that set must contain an open subinterval of every neighborhood of x. In order for f to be qualitatively continuous it must be residual in an open interval about x. Finally then in order for f to be qq–continuous there it must be residual in some open subinterval of every neighborhood of x. Our only applications of this notion will use the observation that if f is qq–continuous at a point x and the point x belongs to $c(A)$ then there is a sequence of points $a_n \in A$ with $a_n \to x$ and $f(a_n) \to f(x)$.

The notion of qq–continuity while formally apparently very weak is equivalent to the property of Baire. We prefer to use the "weaker" version in the statements of the theorems that require some kind of continuity, but it is well to keep in mind that we have not obtained stronger results.

Theorem 6 *A function f has the property of Baire if and only if it is qq–continuous everywhere excepting a set of the first category.*

In fact the proof will show that a function that is qq-continuous at every point of a dense set must have the property of Baire. The notion of qq–continuity suggests that we should require also a similar weakening of the notion of approximate continuity.

Definition 7 A function f is said to be *quasi-approximately continuous* (or simply qa–continuous) at x if

$$\{y : |f(x) - f(y)| < \epsilon\} \cap A \neq \emptyset$$

for every positive number ϵ and for every set A with $x \in d(A)$.

We shall use this terminology since it allows a parallel development between the measure-theoretic and category-theoretic versions of certain theorems. However, again, it should be noted that this formally weaker requirement than approximate continuity is almost everywhere the same as the stronger requirement. A function is measurable if and only if it is almost everywhere approximately continuous; the same is true for qa–continuity.

Theorem 8 *A function f is measurable if and only if it is almost everywhere qa–continuous.*

The proof will show moreover that a function that is qa–continuous at the points of a set of full outer measure must be measurable.

3.2 Local symmetry

There was a query posted in an early issue of this Exchange (see [6]) whether a set (or function) that is everywhere locally symmetric need be measurable. In view of Charzyński's theorem ([4]) this property, which is more severe than the property of that theorem, would require that such a function is even continuous everywhere excepting a scattered set. As Davies [5] and Ruzsa [14] pointed out in the next issue of the Exchange a sharper result than this is possible: such a function is constant off of a closed countable set.

If we ask instead what is the nature of the set of points at which an arbitrary function might be locally symmetric the two covering Lemmas 3 and 4 provide what seems to be a complete analysis. In the theorem we let LS_f denote the set of points at which f is exactly locally symmetric, that is to say the points x at which there is a positive number $\delta(x)$ so that

$$0 < t < \delta(x) \Longrightarrow f(x-t) = f(x+t).$$

We let L_f denote the set of points at which f is constant (a function is constant at a point x if it is constant in a neighborhood of the point x). Evidently the set L_f of points of constancy of f is open and an elementary compactness argument shows that f is constant on each component of that set.

Our theorem is first in a series of results all of which follow from the basic covering theorems and all of which have as their theme a comparison of conditions that hold symmetrically with conditions that hold in an ordinary sense. Thus here we compare a form of "symmetric constancy" with ordinary "constancy"; later we compare symmetric continuity with ordinary continuity, symmetric differentiability with ordinary differentiability, symmetric monotonicity with ordinary monotonicity and so on. In each case the arguments and statements of results are closely related and, as far as possible, we try to preserve the same form of expression and proof in order to show the unity. A single general theorem encompassing all the variants is possible but not very informative.

The theorem asserts that the set $\mathrm{LS}_f \setminus \mathrm{L}_f$ is very small except in what might be considered pathological cases; the example provided in [20, p. 429] shows that these extreme cases can occur. Note in the statement of the theorem that a set that contains no measurable set of positive measure has inner measure zero and a set that contains no second category set with the Baire property is residual in no open interval; the theorem could have been expressed in this language instead.

Theorem 9 *Let f be an arbitrary function, let LS_f denote the set of points at which f is exactly locally symmetric and let L_f denote the set of points at which f is constant. Then the set*

$$LS_f \setminus L_f$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f and qqC_f denote the sets of *points at which f is qa–continuous and qq–continuous respectively then*

$$qaC_f \cap LS_f \setminus L_f$$

has measure zero and

$$qqC_f \cap LS_f \setminus L_f$$

is first category.

As corollaries we have immediately the following observations.

Corollary 10 *Let f be a measurable function. Then the set LS_f of points at which f is exactly locally symmetric is measurable.*

Corollary 11 *Let f be a function with the Baire property. Then the set LS_f of points at which f is exactly locally symmetric has the Baire property.*

Corollary 12 *Let f be a function that is locally symmetric at each point of a measurable set E. Then there is an open set G containing almost every point of E so that f is constant on each component of G.*

Corollary 13 *Let f be a function that is locally symmetric at each point of a set E that has the Baire property. Then there is an open set G containing all but a first category subset of E so that f is constant on each component of G.*

Corollary 14 *Let f be a function that is locally symmetric at each point of a set residual in $\mathbf{R}$. Then f is constant on each component of a dense open set G.*

3.3 Symmetric monotonicity

We turn now to an investigation of Belna, Evans and Humke [2] on the structure of the set of points of symmetric monotonicity. Let SI_f denote the set of points at which the function f is symmetrically nondecreasing and let I_f denote the set of points at which f is nondecreasing in the ordinary sense. That is at points $x \in SI_f$ the relation $f(x-t) \leq f(x+t)$ must hold for all sufficiently small t while at points $x \in I_f$ the relation $f(x') \leq f(x) \leq f(x'')$ holds for $x' < x < x''$ sufficiently close to x. The Borel structure of I_f is easy to obtain; it is of type $\mathcal{G}_{\delta\sigma}$ (see, for example, [18, p. 116]). Belna, Evans and Humke proposed to study the nature of the set $SI_f \setminus I_f$. We obtain the following theorem in our usual fashion from the covering Lemmas 3 and 4. As the proof follows the same lines as that for Theorem 9 it may be omitted.

Theorem 15 *Let f be an arbitrary function, let SI_f denote the set of points at which f is symmetrically nondecreasing and let I_f denote the set of points at which f is nondecreasing in the ordinary sense. Then the set*

$$SI_f \setminus I_f$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f and qqC_f denote the sets of points at which f is qa–continuous and qq–continuous respectively then

$$qaC_f \cap SI_f \setminus I_f$$

has measure zero and

$$qqC_f \cap SI_f \setminus I_f$$

is first category.

From this we immediately obtain the following corollaries. The first is a theorem of Belna, Evans and Humke (as cited above) and the second is a category analogue; their category version uses a different assumption on the function.

Corollary 16 (Belna, Evans and Humke) *If f is measurable then SI_f is measurable and $SI_f \setminus I_f$ has measure zero.*

Corollary 17 *If f has the Baire property then SI_f has the Baire property and $SI_f \setminus I_f$ is first category.*

Thus for well behaved functions the set $SI_f \setminus I_f$ is small both in the sense of measure and of category. For examples illustrating that this cannot be much improved see the article [2].

3.4 Symmetric semi-continuity

We turn now to the problem of determining the continuity properties of functions that are symmetrically continuous. A function f is symmetrically continuous at a point x if

$$\lim_{h\to 0} f(x+h) - f(x-h) = 0.$$

A little more generally f is said to be upper (lower) symmetrically semicontinuous at x if

$$\limsup_{h\to 0} f(x+h) - f(x-h) \le 0 \quad \left(\liminf_{h\to 0} f(x+h) - f(x-h) \ge 0\right).$$

It is symmetrically semicontinuous if it is either upper or lower symmetrically semicontinuous.

The reader may recall that the first result of this type in the literature is the theorem of Fried [8] asserting that everywhere symmetrically continuous functions are continuous at the points of a residual set. The measure-theoretic version is given by the theorem of Stein and Zygmund [16, Lemma 9] showing that, at least for measurable functions, the symmetrically continuous functions are almost everywhere continuous. The measurability assumption was dropped in Preiss [11] in 1971 and Belna [1] in 1983 in related theorems. Finally the complete analysis of the situation was provided by Uher [20] in 1986 who showed that both the measure-theoretic version and the category version follow from the same geometric arguments and that the more natural assumption of symmetric semicontinuity was enough for these results. It was Uher's analysis of this problem that led to the covering lemmas that are our main concern.

Our basic theorem in this section is due to Uher [20] and it combines a measure-theoretic version with a category version by appealing to the covering Lemmas 3 and 4. Note that it is directly analogous to Theorems 9 and 15 both in statement and in proof.

Theorem 18 (Uher) *Let f be an arbitrary function, let SS_f denote the set of points at which f is symmetrically semicontinuous and let C_f denote the set of points which f is continuous in the ordinary sense. Then the set*

$$SS_f \setminus C_f$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f *and* qqC_f *denote the sets of points at which* f *is qa–continuous and qq–continuous respectively then*

$$qaC_f \cap SS_f \setminus C_f$$

has measure zero and

$$qqC_f \cap SS_f \setminus C_f$$

is first category.

This theorem can be restated in somewhat different language; here is a version from Uher [20, Theorem 1, p. 425] which follows from Theorem 18 above both in the measure-theoretic version and in the category version.

Corollary 19 (Uher) *Let* SSD_f *denote the set of points at which a function* f *is not symmetrically semicontinuous. Then* f *is continuous at almost every point of the set* $\mathbf{R} \setminus d(SSD_f)$ *and continuous at all but a first category subset of* $\mathbf{R} \setminus c(SSD_f)$*.*

We now easily draw a number of further corollaries from the main theorem, all again attributable to Uher.

Corollary 20 *A function that has at every point a finite or infinite symmetric derivative is measurable.*

Corollary 21 *Let the function* f *be symmetrically semicontinuous at every point. Then* f *is continuous at every point excepting a set of measure zero and first category. In particular* f *is measurable and has the Baire property.*

Corollary 22 *Let the function* f *be measurable; then* f *is continuous at almost every point at which it is symmetrically semicontinuous.*

Corollary 23 *Let the function* f *have the Baire property; then* f *is continuous at all but a first category subset of the set of points at which it is symmetrically semicontinuous.*

We have too as corollaries the following theorems of Belna [1], Fried [8] and Stein and Zygmund [16] to which we have already alluded.

Corollary 24 (Belna) *Let the function f be symmetrically continuous at almost every point of a measurable set E. Then f is continuous almost everywhere in E.*

Corollary 25 (Fried) *Let the function f be symmetrically continuous on a set residual in $\mathbf{R}$. Then f is continuous at every point excepting a set of first category.*

Corollary 26 (Stein and Zygmund) *Let f be a measurable function and suppose that f is symmetrically continuous at each point x of a measurable set E. Then f is continuous at almost all points of E.*

3.5 Boundedness

While continuous functions are bounded in every compact interval the assumption of symmetric continuity has no such consequence; indeed the function $f(x) = x^{-2}$ is symmetrically continuous (even symmetrically differentiable) but unbounded at the origin. Even so the points of unboundedness of a symmetrically continuous functions cannot be too large as a simple argument shows. The set $E = \{x : \limsup_{y \to x} |f(y)| = \infty\}$ for a symmetrically continuous function f is easily shown to be locally symmetric at every point. Consequently using the characterization of locally symmetric sets mentioned earlier we can prove the following theorem.

Theorem 27 *Let f be an everywhere symmetrically continuous function. Then the set*

$$E = \{x : \limsup_{y \to x} |f(y)| = \infty\}$$

is closed and countable.

We can ask too whether a weaker condition such as

$$|f(x+h) - f(x-h)| = O(1) \quad \text{as} \quad h \to 0$$

requires f to be bounded at any points. By our usual methods the following theorem may be proved. In the theorem we let BS_f denote the set of points at which f is symmetrically bounded, i.e. points x at which

$$\limsup_{h \to 0} |f(x+h) - f(x-h)| < +\infty,$$

and we let B_f denote the set of points at which f is bounded (a function is bounded at a point x if it is bounded in a neighborhood of the point x). Evidently the set B_f of points of boundedness of f is open and an elementary compactness argument shows that f is bounded on each compact subset of that set. For a proof one need only adapt the arguments in the proof of Theorem 9 to this situation.

Theorem 28 *Let f be an arbitrary function, let BS_f denote the set of points at which f is symmetrically bounded and let B_f denote the set of points at which f is bounded. Then the set*

$$BS_f \setminus B_f$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f and qqC_f denote the sets of points at which f is qa–continuous and qq–continuous respectively then

$$qaC_f \cap BS_f \setminus B_f$$

has measure zero and

$$qqC_f \cap BS_f \setminus B_f$$

is first category.

As corollaries we have immediately the following observations.

Corollary 29 *Let f be a measurable function. Then the set BS_f of points at which f is symmetrically bounded is measurable.*

Corollary 30 *Let f be a measurable function. Then f is bounded at almost every point at which it is symmetrically bounded.*

Corollary 31 *Let f be a function possessing the Baire property. Then f is bounded at every point at which it is symmetrically bounded except possibly for a set of the first category.*

3.6 Relations among derivates

There is by now an extensive literature devoted to the study of the relations that hold among the various generalized derivatives. The first such theorem traces back to Levi who showed that while a function can easily have different one sided derivatives at a given point there can be a disagreement between the two one sided derivatives only on a countable set. For a review of some of this literature as it applies to Dini derivatives, approximate Dini derivatives, qualitative derivatives *etc.* see [18, Chapters 6 and 7].

A relation between the symmetric derivates and the ordinary derivates of measurable functions was first established by Khintchine. This was completed by Uher [20, Lemma 2, p. 426]. The following is a version of that theorem with a few refinements, presented in our usual form.

Theorem 32 *Let f be an arbitrary function. Then the set*

$$\{x : \overline{SD}f(x) \neq \overline{D}f(x)\}$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f and qqC_f denote the sets of points at which f is qa–continuous and qq–continuous respectively then

$$qaC_f \cap \{x : \overline{SD}f(x) \neq \overline{D}f(x)\}$$

has measure zero and

$$qqC_f \cap \{x : \overline{SD}f(x) \neq \overline{D}f(x)\}$$

is first category.

As corollaries we immediately have the following assertions. The first two of these are due to Filipczak [7].

Corollary 33 (Filipczak) *Let the function f be measurable. Then the following relations between the extreme symmetric derivates and the ordinary derivates must hold almost everywhere:*

$$\overline{SD}f = \overline{D}f(x) \quad \textit{and} \quad \underline{SD}f(x) = \underline{D}f(x)$$

Corollary 34 (Filipczak) *Let the function f be measurable. Then the extreme symmetric derivates $\overline{SD}f$ and $\underline{SD}f$ are measurable too.*

Corollary 35 *Let the function f have the Baire property. Then the relations between the extreme symmetric derivates and the ordinary derivates*

$$\overline{SD}f = \overline{D}f(x) \quad \text{and} \quad \underline{SD}f(x) = \underline{D}f(x)$$

hold off of a set of the first category.

Finally we should mention that these results provide a perspective on the classical theorem of Khintchine. It is well known (see, for example, Saks [15, p. 230 and p. 234]) that a function f is differentiable at almost every point at which the inequality $\overline{\mathrm{D}}\, f(x) < \infty$ holds. The estimates just obtained show that, at least for measurable functions, the inequality $\overline{\mathrm{D}}\, f(x) < \infty$ is almost everywhere equivalent to the inequality $\overline{\mathrm{SD}}\, f(x) < \infty$ and so the Khintchine theorem follows too as a corollary.

Corollary 36 (Khintchine) *Let f be a measurable function. Then f is differentiable at almost every point x at which*

$$\limsup_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} < +\infty.$$

4 Proofs

In the proofs of the covering lemmas a careful handling of density arguments is needed. This is most easily presented by cataloging the usual properties of density that are used in such proofs. The following properties (taken directly from Uher's presentation in [20]) are readily established for the density notion and are all that are needed in constructing the geometric arguments used in all of the proofs.

Lemma 37 *Let $d(E)$ denote the set of exterior density points of an arbitrary set E defined as*

$$d(E) = \left\{ x : \lim_{h\to 0} \frac{|E \cap (x-h, x+h)|}{2h} = 1 \right\}.$$

Then the following properties hold for all sets of real numbers.
(1) If $A \subset B$ then $d(A) \subset d(B)$.
(2) If $\bigcup A_n = A$ then $d(A) \setminus \bigcup d(A_n)$ has measure zero.
(3) If $A \setminus B$ and $B \setminus A$ both have measure zero then $d(A) = d(B)$.
(4) $d(d(A)) = d(A)$ for any set A.
(5) $A \setminus d(A)$ has measure zero for any set A.
(6) $d(\alpha + \beta A) = \alpha + \beta d(A)$ for any set A and real numbers α and β.
(7) A is measurable if and only if $|d(A) \setminus A| = 0$.
(8) If A is measurable then $d(A \cap B) = d(A) \cap d(B)$ for any set B.
(9) If 0 is a density point for each of the sets A_1, A_2, $\dots A_k$, all but one of which are measurable then for every $p > 0$ there is a positive number δ so that whenever $0 < \tau < \delta$, $(a,b) \subset (-\tau, +\tau)$ with $b - a > \tau/p$ and $\max\{|\alpha_1|, |\alpha_2|, \dots |\alpha_k|\} \leq \tau$ then the set

$$(a,b) \cap \bigcap_{i=1}^{k} (\alpha_i + A_i)$$

has positive measure.

The dual properties for the category notion are identical subject to the correct translation. For "measure zero" read "first category", for "positive measure" read "second category", for "measurable set" read "set with the Baire property" and, of course, replace $d(A)$ by $c(A)$. The above properties now translate directly into the following. By displaying them as identical formally with the density properties it is easy to see how a proof for the density case translates quickly to a proof for the category case.

Lemma 38 *Let $c(E)$ denote the set of second category points of an arbitrary set E defined as*

$$c(E) = \mathbf{R} \setminus \mathit{Closure}\left(\bigcup \{(a,b) : (a,b) \cap E \text{ is first category}\}\right).$$

Then the following properties hold for all sets of real numbers.
(1) If $A \subset B$ then $c(A) \subset c(B)$.
(2) If $\bigcup A_n = A$ then $c(A) \setminus \bigcup c(A_n)$ is first category.
(3) If $A \setminus B$ and $B \setminus A$ are both first category then $c(A) = c(B)$.
(4) $c(c(A)) = c(A)$ for any set A.
(5) $A \setminus c(A)$ is first category for any set A.

(6) $c(\alpha + \beta A) = \alpha + \beta c(A)$ *for any set* A *and real numbers* α *and* β.
(7) A *has the Baire property if and only if* $c(A) \setminus A$ *is first category.*
(8) If A *has the Baire property then* $c(A \cap B) = c(A) \cap c(B)$ *for any set* B.
(9) If 0 *belongs to each of* $c(A_1)$, $c(A_2)$, $\ldots c(A_k)$ *where all but one of the sets* A_1, A_2, $\ldots A_k$ *have the Baire property then for every* $p > 0$ *there is a positive number* δ *so that whenever* $0 < \tau < \delta$, $(a, b) \subset (-\tau, +\tau)$ *with* $b - a > \tau/p$ *and* $\max\{|\alpha_1|, |\alpha_2|, \ldots |\alpha_k|\} \leq \tau$ *then the set*

$$(a, b) \cap \bigcap_{i=1}^{k} (\alpha_i + A_i)$$

is second category.

4.1 Proof of lemma 3

Let us take $b(E) = d(E)$ and give the language of the density proof. (For convenience the single point at which a translation for the $b(E) = c(E)$ proof must be made is indicated.) Also we will shift everything to the origin and assume that the point z is 0. We argue just on the right at 0 as the situation on the left is similarly handled.

We use Lemma 37(9) to choose $\delta < \eta$ so that if $0 < x < \delta$ then the set

$$2b(E) \cap (2E - x) \cap (0, x)$$

is nonempty. This just uses the sets $A_1 = 2b(E)$ and $A_2 = 2E$ and the numbers $\alpha_1 = 0$ and $\alpha_2 = -x$; since $0 \in 2b(E)$ and $0 \in b(2E)$ the property of Lemma 37(9) even shows that the set $2b(E) \cap (2E - x) \cap (0, x)$ can be arranged to have positive measure [to be of the second category]. If $0 < x < \delta$ then we choose

$$u \in b(E) \cap (E - \tfrac{1}{2}x) \cap (0, \tfrac{1}{2}x).$$

Note that $u \in b(E)$ so that, by Lemma 37(6), we know that $0 \in b(2E - 2u)$. Thus if we define

$$A = (2E - 2u) \cap (-2u, 2u)$$

then we have $0 \in b(A)$ as we require.

For any point $a \in A$, $0 < a < x$ then we can exhibit $V : a \rightsquigarrow x$ by a scheme of two reflections

$$a < 2u < x$$

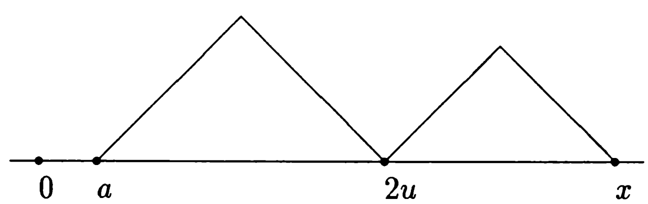

Figure 1: Two reflections in the proof of Lemma 3.

(see Figure 1) and we have only to check that the vertex points are in V. But the midpoint of the first reflection is $(a + 2u)/2 = u + \frac{1}{2}a$ and $a \in A \subset 2E - 2u$ so that $u + \frac{1}{2}a \in E$. Certainly then $(u + \frac{1}{2}a, (2u - a)/2) \in V$ since $(2u - a)/2 < \eta$.

The midpoint of the second reflection is $(x+2u)/2 = u+\frac{1}{2}x$ but by choice $u \in E - \frac{1}{2}x$ so this too is in E and again $(u + \frac{1}{2}x, (x - 2u)/2) \in V$ since $(x - 2u)/2 < \eta$.

4.2 Proof of Lemma 4

The proof is given in the language of the measure-theoretic version but is easily translated for the category version. We simplify by translating to the origin so that we may take $z = 0$. Also it is enough to prove that U may be selected so that $V : z \rightsquigarrow x$ for x on the right of z in U. We suppose there is a set of measure zero N so that every point x of $b(E) \setminus N$ has the local property of the theorem. Because of these local assumptions on V_2 we may by well known methods obtain disjointed sequences of sets $\{E_m^+\}$ and $\{E_m^-\}$ covering the set $b(E) \setminus N$ so that

$$x \in E_m^+,\ 0 < t < 1/m \Longrightarrow (x,t) \in V_2$$

and

$$x \in E_m^-,\ 0 > t > -1/m \Longrightarrow (x,t) \in V_2.$$

We perform in advance the computations necessary to the argument.

Lemma 39 *If $0 \in b(E)$ then a number $\delta > 0$ may be chosen so that for any $0 < x < \delta$ the set $A(x)$ defined as*

$$(2E - x) \cap \left(b(E) + \tfrac{3}{4}x\right) \cap b(E) \cap \left(\tfrac{2}{3}b(E) + \tfrac{1}{2}x\right) \cap \left(\tfrac{1}{2}b(E) + \tfrac{3}{4}x\right) \cap (\tfrac{3}{4}x, x)$$

has positive measure [is second category].

PROOF. In the measure-theoretic version each of the sets in the expression for $A(x)$ has 0 as a density point and all but one of them is measurable; consequently Lemma 37(9) provides a number δ so that this set has positive measure. The category version is similar.

Lemma 40 *If the set $A(x)$ defined as*

$$(2E - x) \cap \left(b(E) + \tfrac{3}{4}x\right) \cap b(E) \cap \left(\tfrac{2}{3}b(E) + \tfrac{1}{2}x\right) \cap \left(\tfrac{1}{2}b(E) + \tfrac{3}{4}x\right) \cap (\tfrac{3}{4}x, x)$$

has positive measure [is second category] then it has a point in common with one of the sets

$$b\left((E + \tfrac{3}{4}x) \cap E_q^+\right)$$

or

$$b\left((E + \tfrac{3}{4}x) \cap E_q^-\right)$$

for some integer q.

PROOF. By definition

$$A(x) \subset \left(b(E) + \tfrac{3}{4}x\right) \cap b(E)$$

and, by Lemma 37(6),

$$b(E) + \tfrac{3}{4}x = b(E + \tfrac{3}{4}x).$$

Together these give

$$A(x) \subset b\left((E + \tfrac{3}{4}x) \cap b(E)\right). \tag{1}$$

Recall that the union $\bigcup_{m=1}^{\infty} E_m^+ \cup E_m^-$ contains almost every point of $b(E)$. Consequently the union

$$\bigcup_{m=1}^{\infty} \left((E + \tfrac{3}{4}x) \cap E_m^+\right) \cup \left((E + \tfrac{3}{4}x) \cap E_m^-\right)$$

contains almost every point of $(E + \frac{3}{4}x) \cap b(E)$. From (1) and Lemma 37(2), we conclude that one of the sets

$$A(x) \cap b\left((E + \tfrac{3}{4}x) \cap E_q^+\right)$$

or

$$A(x) \cap b\left((E + \tfrac{3}{4}x) \cap E_q^-\right)$$

for some integer $q = 1, 2, 3, \ldots$ has positive measure; if not then $A(x)$ has measure zero in contradiction to our assumptions. This completes the proof of the lemma.

The third lemma continues these computations.

Lemma 41 *If y is a point in the set $A(x)$ and*

$$C = \bigcup_{t \in C^-} (t - \kappa(t), t] \cup \bigcup_{t \in C^+} [t, t + \kappa(t))$$

where $\kappa(t) = \min\{\frac{1}{m}, \frac{1}{4}x\}$ for $t \in E_m^+ \cup E_m^-$,

$$C^+ = (E - \tfrac{3}{4}x + \tfrac{1}{2}y) \cap \bigcup_{m=1}^{\infty} E_m^+,$$

and

$$C^- = (E - \tfrac{3}{4}x + \tfrac{1}{2}y) \cap \bigcup_{m=1}^{\infty} E_m^-,$$

then $y \in b(\frac{1}{2}C + \frac{3}{4}x)$.

PROOF. Since the set $\bigcup_{m=1}^{\infty} E_m^+ \cup E_m^-$ contains almost every point of $b(E)$ the set C must contain almost every point of

$$(E - \tfrac{3}{4}x + \tfrac{1}{2}y) \cap b(E).$$

Therefore

$$b(C) \supset b\left((E - \tfrac{3}{4}x + \tfrac{1}{2}y) \cap b(E)\right) = b\left((E - \tfrac{3}{4}x + \tfrac{1}{2}y)\right) \cap b(b(E))$$

by Lemma 37(8). Thus we have from Lemma 37(4) that

$$b(C) \supset \left(b(E) - \tfrac{3}{4}x + \tfrac{1}{2}y\right) \cap b(E). \tag{2}$$

Since by definition $y \in A(x) \subset \frac{1}{2}b(E) + \frac{3}{4}x$ it follows that $2(y - \frac{3}{4}x) \in b(E)$. Similarly the fact that $y \in A(x) \subset \frac{2}{3}b(E) + \frac{1}{2}x$ entails that $\frac{3}{2}(y - \frac{1}{2}x) \in b(E)$. A simple computation now shows

$$2(y - \tfrac{3}{4}x) + \tfrac{3}{4}x - \tfrac{1}{2}y = \tfrac{3}{2}(y - \tfrac{1}{2}x) \in b(E)$$

and hence that $2(y - \frac{3}{4}x) \in b(E) - \frac{3}{4}x + \frac{1}{2}y$. Together with (2) this shows that $2(y - \frac{3}{4}x) \in b(C)$ or equivalently

$$y \in \tfrac{1}{2}b(C) + \tfrac{3}{4}x = b\left(\tfrac{1}{2}C + \tfrac{3}{4}x\right)$$

as required.

The final lemma completes our preliminary computations.

Lemma 42 *If y is a point in the set*

$$A(x) \cap \left(b\left((E + \tfrac{3}{4}x) \cap E_q^+\right) \cup b\left((E + \tfrac{3}{4}x) \cap E_q^-\right)\right)$$

then there is a number β with $1/q > \beta > 0$ so that one of the two sets

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^+ \cap (y - \beta, y)$$

or

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^- \cap (y, y + \beta)$$

is nonempty.

PROOF. By definition $y \in A(x) \subset (\frac{3}{4}x, x)$ so that a positive number β may be chosen smaller than $\min\{\frac{1}{q}, \frac{x}{4}\}$ in such a way that $(y-\beta, y+\beta) \subset (\frac{3}{4}x, x)$.

If y belongs to the set $b\left((E + \frac{3}{4}x) \cap E_q^+\right)$ then, since by Lemma 41 y also belongs to the set $b(\frac{1}{2}C + \frac{3}{4}x)$, we see that y is a density point of both sets $(E+\frac{3}{4}x)\cap E_q^+$ and $\frac{1}{2}C+\frac{3}{4}x$. Since the set C is a union of intervals it is a Borel set (see, for example, [3, p. 57] for a proof) so that in particular $\frac{1}{2}C + \frac{3}{4}x$ is measurable. Consequently the set

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^+ \cap (y - \beta, y)$$

has positive measure. If, on the other hand,

$$y \in b\left((E + \tfrac{3}{4}x) \cap E_q^-\right)$$

a symmetrical proof shows that

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^- \cap (y, y + \beta)$$

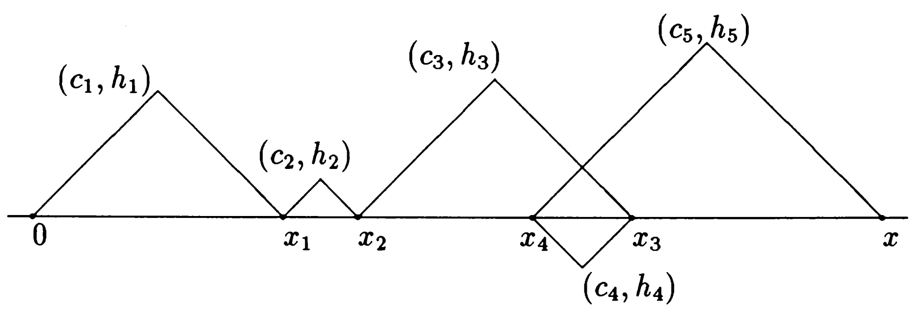

Figure 2: Five reflections in the proof of Lemma 4.

has positive measure. This completes the proof.

Let us now continue the proof of the main theorem. The strategy of the proof is to choose a neighborhood U so that if $x \in U$, $x > 0$ then we may select a sequence

$$0 = x_0, x_1, x_2, x_3, x_4, x_5 = x$$

with centers $c_i = (x_i + x_{i-1})/2$ so the centers c_1, c_3 and c_5 are from E where a forward reflection from V_1 is easily obtained and, more delicately, so that the centers c_2 and c_4 are from the sets E_m^+ or E_m^- where a small forward or backward reflection from V_2 is available. The scheme is exhibited in the figure (Figure 2) where for purposes of illustration the reflection for the c_2 point is forward but the reflection for the c_4 point is backward. All possibilities of course have to be considered.

By Lemma 39 there is a positive number δ which we may select smaller than η so that the set $A(x)$ is nonempty whenever we choose a point $x \in (0, \delta)$. Thus our choice of neighborhood U is evidently to be $(-\delta, \delta)$ and it is clear that δ may depend on E and η as well as the point of density chosen in $b(E)$ (here taken as 0) but is independent of the other variables in the statement of the theorem.

We begin by using Lemma 39 and 40 to select a point $x_4 \in A(x)$ that is also in one of the sets

$$b\left((E + \tfrac{3}{4}x) \cap E_q^+\right)$$

or

$$b\left((E + \tfrac{3}{4}x) \cap E_q^-\right)$$

for some integer q. By the way $A(x)$ is defined the midpoint $c_5 = (x_4+x)/2 \in E$ because $x_4 \in A \subset 2E - x$. Also $0 < x - c_5 < \eta$ so $(c_5, x - c_5) \in V_1$.

Associated with the point $x_4 \in A(x)$ by Lemma 42 there is a number $0 < \beta < 1/q$ so that a point c_4 may be chosen from one of the sets

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^+ \cap (x_4 - \beta, x_4)$$

or

$$(\tfrac{1}{2}C + \tfrac{3}{4}x) \cap (E + \tfrac{3}{4}x) \cap E_q^- \cap (x_4, x_4 + \beta).$$

If c_4 is in the former set then $c_4 \in E_q^+$ and $0 < x_4 - c_4 < \beta < \frac{1}{q}$ so that $(c_4, x_4 - c_4) \in V_2$. On the other hand if c_4 is in the latter set then $c_4 \in E_q^-$ and $0 < c_4 - x_4 < \beta < \frac{1}{q}$ so that again $(c_4, c_4 - x_4) \in V_2$ and in this case the reflection goes backwards (as illustrated in the figure).

It remains only to determine x_1 and c_2 and the whole sequence $0 = x_0, x_1, x_2, x_3, x_4, x_5 = x$ will then be known. We set $x_1 = 2(c_4 - \frac{3}{4}x)$ and we choose a point $c_2 \in C^+$ so that $x_1 \in (c_2 - \kappa(c_2), c_2]$ or alternatively we choose $c_2 \in C^-$ so that $x_1 \in [c_2, (c_2 + \kappa(c_2))$. Such a choice is available: for we have $x_1 = 2(c_4 - \frac{3}{4}x)$ and $c_4 \in (\frac{1}{2}C + \frac{3}{4}x)$ so that x_1 is in C.

This completes the definition of the sequence of reflections that carries 0 to x and we have only to complete the verification that the vertex points in V have been used. We already know the reflections about c_4 and c_5 use points in V_2 and V_1 respectively. For c_1 this is not difficult: recall that $c_1 = (0 + x_1)/2 = c_4 - \frac{3}{4}x$. Since c_4 was chosen in $E + \frac{3}{4}x$ this means that c_1 is in E. But $0 < x_1 - c_1 < \eta$ so the point $(c_1, x_1 - c_1) \in V_1$. For c_2 this is immediate: recall that either $x_1 \in (c_2 - \kappa(c_2), c_2]$ or else $x_1 \in [c_2, c_2 + \kappa(c_2))$ and hence either $(c_2, c_2 - x_1) \in V_2$ or else $(c_2, x_1 - c_2) \in V_2$. Finally for c_3 we have

$$c_3 = \frac{x_2 + x_3}{2} = \frac{(2c_2 - x_1) + (2c_4 - x_4)}{2}$$

and since by definition $x_1 = 2(c_4 - \frac{3}{4}x)$ we have $c_3 = c_2 + \frac{3}{4}x - \frac{1}{2}x_4$. But c_2 by definition belongs to the set $C^+ \cup C^-$ which is a subset of $E - \frac{3}{4}x + \frac{1}{2}x_4$. Consequently $c_3 \in E$ and so $(c_3, x_3 - c_3) \in V_1$.

This completes all the checking and we have produced the sequence of five reflections giving $V : 0 \rightsquigarrow x$. If $-\delta < x < 0$ symmetrical arguments would obtain $V : x \rightsquigarrow 0$ and so the proof is complete.

4.3 Proof of Theorem 6

As for the measurability results we will prove below it is possible to characterize functions with the Baire property in terms of separation properties and for that we shall require some preliminary lemmas. Recall that sets A and B are said to be separated by a set M if $A \subset M$ and $B \cap M = \emptyset$.

Lemma 43 *A function f has the Baire property if and only if for every $a < b$ the sets*

$$U = \{x : f(x) < a\}$$

and

$$V = \{x : f(x) > b\}$$

can be separated by a set which has the Baire property .

PROOF. Certainly the condition is necessary. To show it is sufficient we show, under its assumption, that for any real number c the set of points

$$E_c = \{x : f(x) \leq c\}$$

has the Baire property. For each natural number n choose a set M_n with the Baire property so that

$$M_n \supset \{x : f(x) \leq c\}$$

and

$$M_n \cap \{x : f(x) > c + 1/n\} = \emptyset.$$

It is an easy matter to verify that

$$\bigcap_{n=1}^{\infty} M_n = \{x : f(x) \leq c\}$$

which exhibits the set E_c as having the Baire property as we require.

Lemma 44 *Suppose that the subsets U and V of the reals cannot be separated by a set with the Baire property and that h is a positive function defined on $U \cup V$. Then there are a positive number ε and a nonempty $\mathcal{G}_\delta$ set P of $\mathbf{R}$ with the following properties.*
(a) P is residual in some interval I.
(b) The sets $P \setminus \{x \in U \cap P;\ h(x) > \varepsilon\}$ and $P \setminus \{x \in V \cap P;\ h(x) > \varepsilon\}$ contain no second category set with the Baire property.

PROOF. Just for this proof let us say that sets A and B are simply separated if they can be separated by a set with the Baire property. If for each $k = 1, 2, \ldots$ the sets $\{x \in U;\ h(x) > 1/k\}$ and $\{x \in V;\ h(x) > 1/k\}$ could be separated by a set, say M_k, with the Baire property then U and V would be separated by $\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} M_k$. Hence there is a positive number ε such that the sets $U_1 = \{x \in U;\ h(x) > \epsilon\}$ and $V_1 = \{x \in V;\ h(x) > \epsilon\}$ cannot be separated.

Next we observe that whenever $E_1, E_2, \ldots$ are pairwise disjoint sets with the Baire property such that the sets $A \cap E_i$ and $B \cap E_i$ are separated, say by M_i, then the sets $A \cap \bigcup E_i$ and $B \cap \bigcup E_i$ are separated by $\bigcup (E_i \cap M_i)$. From this we can deduce that there is a set $A \subset \mathbf{R}$ such that A has the Baire property, the sets $U_1 \cap A$ and $V_1 \cap A$ can be separated and that the set $\mathbf{R} \setminus A$ contains no second category set B with the Baire property for which the sets $U_1 \cap B$ and $V_1 \cap B$ can be separated.

To see this choose, if possible, a second category set E_1 with the Baire property so that $U_1 \cap E_1$ and $V_1 \cap E_1$ can be separated. Continue inductively choosing for any ordinal β a second category set

$$E_\beta \subset \mathbf{R} \setminus \bigcup_{\alpha < \beta} E_\alpha$$

with the Baire property so that $U_1 \cap E_\beta$ and $V_1 \cap E_\beta$ can be separated. While the process may continue transfinitely we know that any collection of disjoint second category sets with the Baire property is countable (eg. [10, p. 256]) so that by the observation just made in the preceding paragraph the sets $U_1 \cap \bigcup E_\alpha$ and $V_1 \cap \bigcup E_\alpha$ can be separated.

The set $\mathbf{R} \setminus \bigcup E_\alpha$ has the Baire property and must itself be of second category otherwise U_1 and V_1 can be separated. Thus if we take $A = \bigcup E_\alpha$ then, by its construction, the set $\mathbf{R} \setminus A$ contains no second category set B with the Baire property for which the sets $U_1 \cap B$ and $V_1 \cap B$ can be separated.

We select a set $P \subset A$ so that the assertion 44(a) holds. Clearly P is the required set since the assumption that, for example, the set $P \setminus \{x \in U \cap P;\ h(x) > \epsilon\}$ contains a second category set B with the Baire property immediately implies that the sets $\{x \in U \cap B;\ h(x) > \epsilon\}$ and $\{x \in V \cap B;\ h(x) > \epsilon\}$ are separated by B.

We now complete the proof of Theorem 6. Recall ([10, p. 400]) that a function f has the Baire property if and only if there is a set Z residual in

$\mathbf{R}$ so that f is continuous relative to Z. Thus a function f that has the property of Baire is certainly qq–continuous everywhere off of a set of the first category. Suppose then to prove the converse that f is qq–continuous except at the points of a set N of first category. In view of Lemma 43 it is enough to show that for every $a < b$ the sets

$$U = \{x : f(x) < a\}$$

and

$$V = \{x : f(x) > b\}$$

can be separated by a set having the Baire property.

If not then by Lemma 44 there is a set P residual in an interval I so that U and V are full in P in a category sense. Since f is qq–continuous at every point $x \in U \setminus N$ each of these points has the property that $x \notin c(V)$ since that would require $U \cap V \neq \emptyset$; for the same reason every point $x \in V \setminus N$ has the property that $x \notin c(U)$. But this is impossible for then any point of category density p of P, which is then also a point of density for both U and V since these sets are full in P, can belong to neither U nor V. But P is residual in I which provides our contradiction.

Thus U and V can be separated by a set with the Baire property for all $a < b$ and so f has the Baire property as required.

4.4 A simpler proof of Theorem 6

The proof just given for Theorem 6 has the advantage of being easy to translate for the measure-theoretic version (Theorem 8). This preserves a degree of symmetry between the measure theory and category results. Unfortunately efforts to preserve such relations often can become forced, and here is no exception. The referee has a supplied a more elegant and revealing proof that directly addresses the category situation. We present this here.

For any function f and any set H let $\omega(f, H, x)$ denote the oscillation of the restriction of the function f to the set H taken at the point x. If f is qq-continuous at the points of a residual set (or even just a dense set) then we shall show that f has the Baire property. It is evidently enough to show that for every integer n there is a residual set A_n such that $\omega(f, A_n, x) < 1/n$ for every point $x \in A_n$. In this case f is continuous relative to the residual set $\bigcap_{n=1}^{\infty} A_n$ and so has the Baire property.

Let n be fixed. Let $\mathcal{H}$ denote the set of pairs (I, B) where I is an open interval, $B \subset I$, B is residual in I and $\omega(f, B, x) < 1/n$ for every $x \in B$. Then for every interval J there is a pair $(I, B) \in \mathcal{H}$ such that $I \subset J$. Indeed if $x \in J$ is a point of qq-continuity of f then the set

$$C = \{y : |f(x) - f(y)| < 1/(2n)\}$$

must be residual in some open subinterval I of J and then $(I, I \cap C) \in \mathcal{H}$. This implies that there is a sequence $(I_k, B_k) \in \mathcal{H}$ such that $I_k \cap I_m = \emptyset$ for $k \neq m$ and $\bigcup_{k=1}^{\infty} I_k$ is everywhere dense in $\mathbf{R}$. Thus the set $A_n = \bigcup_{k=1}^{\infty} B_k$ is residual and $\omega(f, A_n, x) < 1/n$ holds for every $x \in A_n$. This completes the proof.

Note that this proof shows that if f is qq-continuous at the points of a dense set it must have the Baire property.

4.5 Proof of Theorem 8

It is evident that a measurable function, since it is almost everywhere approximately continuous, must be almost everywhere qa–continuous. Thus it is enough to show that a function that is almost everywhere qa–continuous is measurable.

We shall obtain our measurability results as separation properties and for that we shall require the lemmas of this section. Recall that sets A and B are said to be separated by a set M if $A \subset M$ and $B \cap M = \emptyset$. The first lemma is just the elementary and well known observation that measurability may be characterized as a separation property.

Lemma 45 *A function f is measurable if and only if for every $a < b$ the sets*

$$U = \{x : f(x) < a\}$$

and

$$V = \{x : f(x) > b\}$$

can be separated by a measurable set.

PROOF. This is identical to Lemma 43.

The next lemma and its proof are reproduced from Preiss and Thomson [12]. This gives a useful criterion of measurability in light of Lemma 45.

Lemma 46 *Suppose that the subsets U and V of the reals cannot be separated by a measurable set and that h is a positive function defined on $U \cup V$. Then there are a positive number ε and a nonempty compact subset P of $\mathbf{R}$ with the following properties.*
(a) The intersection $I \cap P$ has positive measure whenever I is an open interval meeting P.
(b) The sets $\{x \in U \cap P;\ h(x) > \varepsilon\}$ and $\{x \in V \cap P;\ h(x) > \varepsilon\}$ are both of full outer measure in P.

PROOF. If for each $k = 1, 2, \ldots$ the sets $\{x \in U;\ h(x) > 1/k\}$ and $\{x \in V;\ h(x) > 1/k\}$ could be separated by measurable sets, say M_k, then U and V would be separated by $\bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} M_k$. Hence there is a positive number ε such that the sets $\{x \in U;\ h(x) > \epsilon\}$ and $\{x \in V;\ h(x) > \epsilon\}$ cannot be separated by a measurable set.

Next we observe that whenever $E_1, E_2, \ldots$ are pairwise disjoint measurable sets such that the sets $A \cap E_i$ and $B \cap E_i$ are separated by measurable sets, say M_i, then the sets $A \cap \bigcup E_i$ and $B \cap \bigcup E_i$ are separated by $\bigcup (E_i \cap M_i)$. From this we easily deduce that there is a measurable set $A \subset \mathbf{R}$ such that the sets $\{x \in U \cap A;\ h(x) > \epsilon\}$ and $\{x \in V \cap A;\ h(x) > \epsilon\}$ can be separated by a measurable set and that the set $\mathbf{R} \setminus A$ contains no measurable set B of positive measure for which the sets $\{x \in U \cap B;\ h(x) > \epsilon\}$ and $\{x \in V \cap B;\ h(x) > \epsilon\}$ would be separated by a measurable set.

Since the set $\mathbf{R} \setminus A$ has positive measure it contains a nonempty compact set P for which the assertion 46(a) holds. Clearly P is the required set since the assumption that, for example, the set $P \setminus \{x \in U \cap P;\ h(x) > \epsilon\}$ contains a measurable set B of positive measure immediately implies that the sets $\{x \in U \cap B;\ h(x) > \epsilon\}$ and $\{x \in V \cap B;\ h(x) > \epsilon\}$ are separated by B.

We may now complete the proof of Theorem 8. Suppose that f is almost everywhere qa–continuous. It is enough to show that for every $a < b$ the sets

$$U = \{x : f(x) < a\}$$

and

$$V = \{x : f(x) > b\}$$

can be separated by a measurable set. If not then by Lemma 46 there is a measurable set P of positive measure so that U and V are of full outer measure in P. Since f is almost everywhere qa–continuous almost every point

x in U has the property that $x \notin d(V)$ since that would require $U \cap V \neq \emptyset$; for the same reason almost every point x in V has the property that $x \notin d(U)$. But this is impossible for then any point of density p of P, which is then also a point of density for both U and V since these sets are full in P, can belong to neither U nor V. But almost every point of P is a point of density which provides our contradiction.

Thus U and V can be separated by a measurable set for all $a < b$ and so f is measurable. This completes the proof.

4.6 Proof of Theorem 9

It is enough to indicate the measure-theoretic version as the translation to the category version is easily carried out. As usual $b(E)$ denotes the set of density points $d(E)$ in the former version and the second category points $c(E)$ in the latter.

Let E be a subset of $\mathrm{LS}_f \setminus \mathrm{L}_f$ and let

$$V = \{(x,t) : x \in E,\ \ f(x-t) = f(x+t)\}.$$

By the assumptions in the theorem at each point $x \in E$ there is a positive number $\delta(x)$ so that

$$0 < |t| < \delta(x) \Longrightarrow (x,t) \in V.$$

We firstly obtain, by standard methods, a partition $\{E_n\}$ of E so that

$$x \in E_n,\ 0 < |t| < 1/n \Longrightarrow (x,t) \in V.$$

For the first part of the theorem we assume that E is measurable. We may apply the covering Lemma 4 with $V_1 = V_2 = V$ to the set $b(E_n)$ for under this measurability assumption on the set E almost every point of $b(E_n)$ is contained in E and so, for almost every point x in $b(E_n)$,

$$0 < |t| < \delta(x) \Longrightarrow (x,t) \in V.$$

Thus we obtain that for any $z \in b(E_n)$ there is a neighborhood U of z so that, for any $x \in U$,

$$z < x \Longrightarrow V : z \rightsquigarrow x$$

and

$$x < z \Longrightarrow V : x \rightsquigarrow z$$

by five reflections in V. But each pair $(s,t) \in V$ has $f(s+t) = f(s-t)$ and this easily establishes that $f(x) = f(z)$ for any such $x \in U$.

Consequently each point $z \in b(E_n)$ is a point of constancy of f. Accordingly $b(E_n) \subset \mathrm{L}_f$ and so $E \cap b(E_n) = \emptyset$ for each n. But $E = \bigcup_{n=1}^{\infty} E_n$ so that $b(E) \setminus \bigcup_{n=1}^{\infty} b(E_n)$ has measure zero. But we also know that $E \setminus b(E)$ has measure zero and from this it follows that E itself must have measure zero as required.

We turn now to the second part of the theorem and drop the assumption that E is measurable. Take $E = \mathrm{LS}_f \setminus \mathrm{L}_f$. Let z be a point in both $b(E_n)$ and in qaC_f. We have the conditions to apply the covering Lemma 3; therefore there is a positive number δ so that for every point $0 < |z - x| < \delta$ there is a set A_x having z as a point of density and for all $a \in A_x$,

$$z < a < x \Longrightarrow V : a \rightsquigarrow x$$

and

$$z > a > x \Longrightarrow V : x \rightsquigarrow a$$

by two reflections in V.

Once again this gives $f(a) = f(x)$ for such points. But z is a point of qa–continuity of f so that, since A_x has density 1 at z some sequence of points a_n in A_x can be found with $f(a_n) \to f(z)$. From this we conclude that $f(z) = f(x)$ for all $|z - x| < \delta$ so that, again, z is a point of constancy of f. Thus $\mathrm{qaC}_f \cap b(E_n) \subset \mathrm{L}_f$ and so $\mathrm{qaC}_f \cap E \cap b(E_n) = \emptyset$ for each n. As before then we conclude that $E \cap \mathrm{qaC}_f$ itself must have measure zero as required.

4.7 Proof of Theorem 18

As before it is enough to indicate the measure-theoretic version as the translation to the category version is easily carried out and again $b(E)$ denotes the set of density points in the former version and the second category points in the latter. The proof is essentially that of Uher but rephrased to accommodate the covering language promoted here.

Let E be a subset of $\mathrm{SS}_f \setminus \mathrm{C}_f$, let

$$V(\epsilon) = \{(x,t) : x \in E,\ f(x+t) - f(x-t) < \epsilon\},$$

let E^+ denote the set of points in E at which f is upper semicontinuous and let E^- denote the set of points there at which f is lower semicontinuous. By the assumptions in the theorem at each point $x \in E$ there is a positive number $\delta(x)$ so that $V(\epsilon)$ has the properties

$$x \in E^+, 0 < t < \delta(x) \Longrightarrow (x,t) \in V(\epsilon)$$

and

$$x \in E^-, 0 > t > -\delta(x) \Longrightarrow (x,t) \in V(\epsilon).$$

As usual we partition the set E^+ using the collection $V(1/m)$ to obtain a partition $\{E^+_{nm}\}$ of that set with the properties that

$$x \in E^+_{nm}, 0 < t < 1/n \Longrightarrow (x,t) \in V(1/m).$$

For the first part of the theorem if E is measurable then we may apply the covering Lemma 4 with $V_1 = V_2 = V(1/m)$ to the set $b(E^+_{mn})$ for, under the measurability assumption, almost every point of $b(E^+_{mn})$ is contained in E and so for almost every point x in $b(E^+_{mn})$ either

$$0 < t < \delta(x) \Longrightarrow (x,t) \in V(1/m).$$

or

$$0 > t > \delta(x) \Longrightarrow (x,t) \in V(1/m).$$

Thus we obtain that for any $z \in b(E^+_{mn})$ there is a neighborhood U of z so that, for any $x \in U$,

$$z < x \Longrightarrow V : z \rightsquigarrow x$$

and

$$x < z \Longrightarrow V : x \rightsquigarrow z$$

by five reflections in $V(1/m)$. But each pair $(s,t) \in V$ has $f(s+t) - f(s-t) < 1/m$ and this gives $f(z) - f(x) < 5/m$ and $f(y) - f(z) < 5/m$ for all $x < z < y$ in U. Consequently

$$\liminf_{x \to z-} f(x) \geq f(z) - 5/m$$

and

$$\limsup_{y \to z+} f(y) \leq f(z) + 5/m$$

for all z in $\bigcup_{n=1}^{\infty} b(E_{mn}^+)$. From this it follows that

$$\liminf_{x \to z-} f(x) \geq f(z)$$

and

$$\limsup_{y \to z+} f(y) \leq f(z)$$

for all

$$z \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} b(E_{mn}^+).$$

If we apply the same arguments to the set E^- we will have a parallel situation on the set

$$z \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} b(E_{mn}^-).$$

By a well known theorem of W. H. Young (see [18, p. 53]) this means f is continuous at every point of

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} b(E_{mn}^+) \cup b(E_{mn}^-)$$

with at most countably many exceptions.

Thus nearly every point of this set A is a point of continuity of f and therefore $A \subset \mathrm{C}_f \cup N$ for some countable set N and so $E \cap A$ has measure zero. But $E = \bigcup_{n=1}^{\infty} E_{mn}^+ \cup E_{mn}^-$ for each m so that

$$b(E) \setminus \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} b(E_{mn}^+) \cup b(E_{mn}^-)$$

has measure zero. But we also have that $E \setminus b(E)$ has measure zero and from this it follows that E itself must have measure zero as required.

We turn now to the second part of the theorem. We drop the assumption that E is measurable and take $E = \mathrm{SS}_f \setminus \mathrm{C}_f$. Let z be a point in both $b(E_{mn}^+)$ and in qaC_f. We have the conditions to apply the covering Lemma 3; therefore there is a positive number δ so that for every point $0 < |z - x| < \delta$ there is a set A_x having z as a point of density and for all $a, b \in A_x$,

$$z < a < x \Longrightarrow V(1/m) : a \rightsquigarrow x$$

and

$$z > b > x \Longrightarrow V(1/m) : x \rightsquigarrow a$$

by two reflections in $V(1/m)$.

Once again this gives $f(x) - f(a) < 2/m$ and $f(b) - f(x) < 2/m$ for all such points. But z is a point of qa–continuity of f so that since A_x has density 1 at z some sequences of points $a_n > z > b_n$ in A_x can be found with $f(a_n) \to f(z)$ and $f(b_n) \to f(z)$. Thus in almost the same manner as before we obtain

$$\liminf_{x \to z-} f(x) \geq f(z)$$

and

$$\limsup_{y \to z+} f(y) \leq f(z)$$

for all

$$z \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} b(E^{+}_{mn}).$$

Again we argue using the set E^{-} in the obvious manner, appeal to the theorem of Young again and we can conclude that the set $E \cap \mathrm{qaC}_f$ itself must have measure zero as required.

4.8 Proof of Theorem 32

For the proof we notice that the relation $\overline{\mathrm{SD}}\, f(x) \leq \overline{\mathrm{D}}\, f(x)$ must hold at every point; consequently the set of points where $\overline{\mathrm{SD}}\, f(x) \neq \overline{\mathrm{D}}\, f(x)$ can be written as a countable union

$$\bigcup_{r \in Q} \{x : \overline{\mathrm{SD}}\, f(x) < r < \overline{\mathrm{D}}\, f(x)\}$$

where the union is over all rational numbers r. Therefore the theorem is proved by showing that each set of this form has the required property. This leads us to the following lemma; once we have proved this we will have completed the proof of the theorem.

Lemma 47 *Let f be an arbitrary function and K a real number. Then the set*

$$\{x : \overline{SD}\, f(x) < K\} \setminus \{x : \overline{D}\, f(x) \leq K\}$$

contains no measurable set of positive measure and no second category set having the Baire property. If, furthermore, qaC_f and qqC_f denote the sets of points at which f is qa–continuous and qq–continuous respectively then

$$qaC_f \cap \{x : \overline{SD} f(x) < K\} \setminus \{x : \overline{D} f(x) \leq K\}$$

has measure zero and

$$qqC_f \cap \{x : \overline{SD} f(x) < K\} \setminus \{x : \overline{D} f(x) \leq K\}$$

is first category.

PROOF. Again we indicate only the measure-theoretic version with $b(E)$ denoting the set of density points. Let E be a subset of

$$\{x : \overline{\mathrm{SD}}\, f(x) < K\} \setminus \{x : \overline{\mathrm{D}}\, f(x) \leq K\}$$

and let

$$V = \{(x,t) : x \in E, f(x+t) - f(x-t) < 2Kt\}.$$

By the assumptions in the theorem, at each point $x \in E$ there is a positive number $\delta(x)$ so that V has the properties

$$x \in E,\ 0 < t < \delta(x) \Longrightarrow (x,t) \in V.$$

As usual we obtain first a partition $\{E_n\}$ of the set E with the properties that

$$x \in E_n,\ 0 < t < 1/n \Longrightarrow (x,t) \in V.$$

For the first part of the theorem if E is measurable then we may apply the covering Lemma 4 with $V_1 = V_2 = V$ to the set $b(E_n)$ for, under the measurability assumption, almost every point of $b(E_n)$ is contained in E and so for almost every point x in $b(E_n)$

$$0 < t < \delta(x) \Longrightarrow (x,t) \in V.$$

Thus we obtain that for any $z \in b(E_n)$ there is a neighborhood U of z so that, for any $x \in U$,

$$z < x \Longrightarrow V : z \rightsquigarrow x$$

and

$$x < z \Longrightarrow V : x \rightsquigarrow z$$

by five reflections in V. But each pair $(s,t) \in V$ has $f(s+t)-f(s-t) < 2Kt$ and this gives $f(z) - f(x) < K(z-x)$ and $f(y) - f(z) < K(y-z)$ for all $x < z < y$ in U. Consequently

$$\overline{\mathrm{D}}\, f(z) \leq K$$

at each such point.

Thus the set $b(E_n)$ is contained in the set $\{x : \overline{\mathrm{D}}\, f(x) \leq K\}$ while E is disjoint from that set. But $E = \bigcup_{n=1}^{\infty} E_n$ so that

$$b(E) \setminus \bigcup_{n=1}^{\infty} b(E_n)$$

has measure zero. But we also have that $E \setminus b(E)$ has measure zero and from this it follows that E itself must have measure zero as required.

We turn now to the second part of the theorem. We drop the assumption that E is measurable and take E as the set

$$\{x : \overline{\mathrm{SD}}\, f(x) < K\} \setminus \{x : \overline{\mathrm{D}}\, f(x) < K\}.$$

Let z be a point in both $b(E_n)$ and in qaC_f. We have the conditions to apply the covering Lemma 3; therefore there is a positive number δ so that for every point $0 < |z - x| < \delta$ there is a set A_x having z as a point of density and for all $a, b \in A_x$,

$$a < x < z \Longrightarrow V : a \rightsquigarrow x$$

and

$$z < b < y \Longrightarrow V : b \rightsquigarrow y$$

by two reflections in V.

Once again this gives $f(x) - f(a) < K(x-a)$ and $f(y) - f(b) < K(y-b)$ for such points. But z is a point of qa–continuity of f so that since A_x has density 1 at z some sequences of points $a_n > z > b_n$ in A_x can be found with $f(a_n) \to f(z)$ and $f(b_n) \to f(z)$. Thus in almost the same manner as before we obtain

$$\overline{\mathrm{D}}\, f(z) \leq K$$

at each such point so we conclude in the usual fashion that the set $E \cap \mathrm{qaC}_f$ itself must have measure zero as required.

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