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THROWING A DART AT FREILING'S ARGUMENT AGAINST THE CONTINUUM HYPOTHESIS

In his article "Axioms of Symmetry: Throwing Darts at the Real Number Line," Freiling [3] purports to "give a simple philosophical 'proof' of the negation of Cantor's continuum hypothesis (CH)." The purpose of the present note is to show mathematically why Freiling's argument is not persuasive.

Freiling proceeds as follows in his case against CH:

"Suppose we were to throw a random dart at the real number line (i.e., the interval [0,1]) and ask whether the dart landed on a rational number. The outcome is, of course, predictable. We could say in advance that the dart will (with probability one) land on an irrational number. Furthermore, let us agree that the reason does not depend on any particular property of the set of rational numbers except that it is countable and its members are determined before we make our throw.

"Now suppose we were to throw two random darts and ask whether the second dart was a rational multiple of the first one. The answer would likewise be no, since by the time we throw the second dart there are only countably many points which it has to miss, and membership in this countable set is predetermined by the first dart.

"Suppose then that we have a function $f: R \to R_{\aleph_0}$ (i.e., f assigns to each real a countable set of reals). The second dart will not be in the countable set assigned to the first dart. Now, by the symmetry of the situation (the real number line does not really know which dart was thrown first or second), we could also say that the first dart will not be in the set assigned to the second one. This leads us to the following natural proposition:

$$A_{\aleph_0}. \qquad \forall f: R \to R_{\aleph_0} (\exists x_1 x_2) x_2 \notin f(x_1) \wedge x_1 \notin f(x_2),$$

the intuition being that x_1 and x_2 could be found by independently throwing two random darts."

Freiling then proves that $A_{\aleph_0}$ is equivalent to $\neg$CH. We shall return to this, but we note that he also considers, among others, the following propositions (in the second of which "null" refers to Lebesgue measure zero) from which he derives what could again be considered surprising conclusions:

$$A_{<2^{\aleph_0}}. \qquad \forall f: R \to R_{<2^{\aleph_0}} (\exists x_1 x_2) x_2 \notin f(x_1) \wedge x_1 \notin f(x_2),$$

$$A_{null}. \qquad \forall f: R \to R_{null} (\exists x_1 x_2) x_1 \notin f(x_2) \wedge x_2 \notin f(x_1).$$

In each of these propositions, the set that the thrown dart has to miss is small in some cardinal or measure-theoretic sense and that is avowedly all that Freiling bases his probability argument and intuition upon.

Consider now the following proposition, in which "n.d." stands for "nowhere dense":

$$A_{n.d.} . \quad \forall f: R \rightarrow R_{n.d.} \; (\exists x_1 x_2) x_2 \notin f(x_1) \wedge x_1 \notin f(x_2).$$

If, for example, we consider a countable nowhere dense set, then such a set is just as small in a cardinal sense as the set of rational numbers, and if Freiling's intuition is correct, we should expect a startling conclusion to follow from $A_{n.d.}$.

But no such conclusion is forthcoming, because $A_{n.d.}$ is true: it expresses a theorem in the theory of independent sets [2], namely, that if the "picture" associated with every real x is a nowhere dense set, then there exists an independent pair; in fact, there is not only an independent pair, but indeed a countable independent set [2], and even an everywhere dense independent set [1].

Wherein lies the fallacy? It is in asking us to "agree that the reason does not depend on any particular property of the set of rational numbers except that it is countable," for it is precisely the distribution of the rational numbers in the unit interval that is decisive: replace the set of rational numbers with a countable nowhere dense set, and there is no case against CH.

Intuition and probability arguments can be pitfalls, especially in set theory.

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