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An Answer to a Question of R.G. Gibson and F. Roush

Theorem. There exists a connected function $f : [0,1] \to R$ which is not almost continuous.

Proof. Let C be the Cantor set in the interval $I = [0,1]$, $\{(a_n, b_n) : n \in \{1,2,\ldots\}\}$ the set of all components of $I \backslash C$, and $f : I \to R$ the function defined by

$$f(x) = \begin{cases} 2 \cdot \frac{x-a_n}{b_n-a_n} - 1 & \text{for } x \in [a_n, b_n] \\ 0 & \text{otherwise.} \end{cases}$$

The graph G_f of f is connected in R^2. To see this let us observe that f is continuous at every point $x \in \bigcup_{n=1}^{\infty} (a_n, b_n)$ and at a_n from the right and at b_n from the left. The cluster set of f at a_n from the left equals $[-1,1]$. If K is any continuum with a non-degenerated projection on the x-axis such that $\text{proj}_x K = [a_n - \delta, a_n]$ for some $\delta > 0$, $\text{proj}_y K \subset (-1,1)$, then $K \cap G_f \neq \phi$ since G_f contains a sequence of segments with endpoints $(a_{n_k}, -1)$, $(b_{n_k}, 1)$ for some sequence (a_{n_k}) such that $a_{n_k} \to a_n$, $a_{n_k} < a_n$. The same argument shows that f is connected at the other points of the Cantor set. Hence f is connected.

We shall show that f is not almost continuous. To this end denote by A_n the open parallelogram with the vertices at the points $(a_n - \frac{1}{10}(b_n - a_n), -1)$, $(a_n, -\frac{11}{10})$, $(b_n, \frac{11}{10})$ and $(b_n + \frac{1}{10}(b_n - a_n), 1)$, and by (a_{n_0}, b_{n_0}) a fixed component of $I \backslash C$. There is a subsequence $(a_{n_k})_{k \geq 1}$ of the sequence $(a_n)_{n \geq n_0}$ such that $b_{n_k} < a_{n_{k+1}}$ and $a_{n_k} \to a_{n_0}$ monotonically.

Set

$$U = (-\frac{1}{10}, a_{n_1}) \cup \bigcup_{k=1}^{\infty} (b_{n_k}, a_{n_{k+1}}) \cup (a_{n_0}, \frac{11}{10})$$

and note that the open set

$$G = \bigcup_{n=1}^{\infty} A_n \cup \left[U \times \left(-\frac{1}{10}, \frac{1}{10} \right) \right]$$

contains G_f. Our purpose is to show that there is no continuous function whose graph is included in G. Suppose that the graph G_g of some function $g : I \to R$ is contained in G. Then g takes values less than $-\frac{8}{10}$ and values greater than $\frac{8}{10}$ on each interval (a_{n_k}, b_{n_k}). Hence the cluster set of g at a_{n_0} from the left contains the interval $[-\frac{8}{10}, \frac{8}{10}]$. Therefore, g cannot be continuous at $x = a_{n_0}$ and so f cannot be almost continuous.

As a corollary we can obtain an answer to a question posed in [2], p. 258 (also see [1], p. 441):

Corollary. There exists a connectivity function $f : I \to I$ which is not the uniform limit of a sequence of almost continuous functions $f_m : I \to I$.

Suppose that the function f in the preceding proof is the uniform limit of a sequence of almost continuous functions $f_m : I \to I$. Then there is an m_0 such that

$$| f_{m_0}(x) - f(x) | < \frac{1}{10} \text{ for all } x \text{ in } I.$$

Hence $G_{f_{m_0}} \subset G$. But f_{m_0} is almost continuous, so that G must contain a continuous function and we arrive at a contradiction.

References

[1] A.M. Bruckner and J. Ceder, "On jumping functions by connected sets", Czech. Math. J. 22, 1972, 435-448.

[2] R.G. Gibson and F. Roush, "The uniform limit of connectivity functions", Real Analysis Exchange 11, 1985-86, 254-259.

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