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ON DISCONTINUITY POINTS FOR CLOSED GRAPH FUNCTIONS

We say that a function f from a space X into a space Y has a closed graph if the graph of the function f, i.e. the set $\{(x,y) \in X \times Y; y = f(x)\}$ is a closed subset of the product $X \times Y$. We denote by C_f (D_f) the set of all points at which the function f is continuous (discontinuous).

There are many papers which deal with the set D_f for closed graph functions. (See for example [1], [2] or [4].) The purpose of the present paper is to continue the investigation of this set.

Proposition A. (See [4].) Let $I \subset R$ be an interval. Then for each closed graph function $f : I \to R$ the set D_f is closed and nowhere dense.

Proposition B. (See [1].) Let $f : X \to R^n$ have a closed graph, where X is a Hausdorff space. Let $x \in D_f$. Then f is unbounded in every neighborhood of the point x.

Theorem 1. Let $f : I \to R$ have a closed graph, where $I \subset R$ is an interval. Let $x \in D_f$. Then for each neighborhood U of x there is an interval $J \subset U \cap C_f$ such that f is unbounded on J.

Proof. Suppose to the contrary that there is a $\delta > 0$ such that for each interval $J \subset (x - \delta, x + \delta) \cap I \cap C_f$ the function f is bounded on J. Put $F = [x - \delta/2, x + \delta/2] \cap I \cap D_f$. Since f is a Baire class one function (See [4].), there is an $x_0 \in F$ such that the function $f|_F$ is continuous at x_0. Put $V = (x - \delta, x + \delta) \cap I \cap C_f$. Since V is open in I, there is a countable family J of pairwise disjoint open intervals such that $V = \bigcup J$. Since $x_0 \in D_f$, the function f is unbounded in each neighborhood of x_0. Thus there is a monotone sequence $\{x_n\}$ of points $x_n \in U$ such that $x_n \to x_0$ and the sequence $\{f(x_n)\}$ is unbounded. Suppose that $x_n < x_0$ for each $n = 1, 2, \ldots$. (The opposite case is similar.) Then for each n there is a $J_n \in J$ such that $x_n \in J_n$. Let $J_n = (a_n, b_n)$. Then $x_n < b_n \leq x_0$ for each $n = 1, 2, \ldots$. Since f has a closed graph and it is by assumption bounded on each J_n, the function $f|_{\overline{J_n}}$ is continuous. Since $f|_F$ is continuous at x_0, it follows that $f(b_n) \to f(x_0)$. From the Darboux property it follows that f assumes any value lying between $f(x_n)$ and $f(b_n)$ at least once on J_n $(n = 1, 2, \ldots)$, which contradicts the closedness of the graph of f.

Definition. (See [3].) A function f defined on a topological space X with range in a topological space Y is said to be quasicontinuous at the point $x \in X$ if for any neighborhood U of the point x and any neighborhood V of $f(x)$ there is an open set $\emptyset \neq G \subset U$ such that $f(G) \subset V$. A function f is said to be quasicontinuous if it is quasicontinuous at each point $x \in X$.

Note that if a function $h : R \to R$ is such that $h(x) = \sin(1/x)$ for $x \neq 0$, then h is quasicontinuous if and only if $-1 \leq h(0) \leq 1$; that is, there is a closed graph function $f : R \to R$ such that $h(x) = \sin(f(x))$ for each $x \in R$. The sufficiency of this condition is true in general as the following theorem shows.

Theorem 2. Let $I \subset R$ be an interval. Let $f : I \to R$ have a closed graph. Then the composite function $h = \sin(f)$ is quasicontinuous.

Proof. Quasicontinuity at the continuity points of f is evident. Suppose that $x \in D_f$. Let V be an open neighborhood of the point $h(x) = \sin(f(x))$. From the continuity of sin it follows that the set $\sin^{-1}(V)$ is open. Since sin is periodic, there is an open interval (a, b) such that $(a + 2k\pi, b + 2k\pi) \subset \sin^{-1}(V)$ for each integer k. Let $\delta > 0$. Since $x \in D_f$, by Theorem 1 there is an interval $J \subset (x - \delta, x + \delta) \cap I \cap C_f$ such that f is unbounded on J. Suppose that f is unbounded below on J. (The opposite case is similar.) Let $x_0 \in J$ be arbitrary. Let k_0 be an integer such that $f(x_0) < a + 2k_0\pi$. From the Darboux property it follows that there is $w \in J$ such that $f(w) \in (a + 2k_0\pi, b + 2k_0\pi)$. Since $w \in C_f$, there is an interval $G \subset J$ such that $f(G) \subset (a+2k_0\pi, b+2k_0\pi)$. Thus $h(G) \subset V$. This shows that h is quasicontinuous at the point x.

The following example shows that the assumption, "I is an interval" in Theorem 2 cannot be replaced by the assumption "I is a subset of R".

Example. Let $Q = \{q_1, q_2, \ldots\}$ be a countable, dense subset of R. Let $f : Q \to R$, $f(q_n) = n\pi/2$ $(n = 1, 2, \ldots)$. Then f has a closed graph, but $\sin(f)$ is not quasicontinuous.

By the preceding methods it is not difficult to verify (ii) implies (i) of the following theorem.

Theorem 3. Let $g : R \to R$ be continuous. Then the following statements are equivalent:

(i) for each closed graph function $f : R \to R$ the composite function $g(f)$ is quasicontinuous,

(ii) for each open set V in R such that $g^{-1}(V) \neq \emptyset$, $\sup g^{-1}(V) = \infty$ and $\inf g^{-1}(V) = -\infty$.

Proof of (i) implies (ii). Deny. Suppose that there is an open set V in R such that $g^{-1}(V) \neq \emptyset$ and $\sup g^{-1}(V) < \infty$. (The second case is similar.) Let $y \in g^{-1}(V)$ be arbitrary. Let $f : R \to R$, $f(0) = y$, $f(x) = 1/\mid x \mid + \sup g^{-1}(V)$ otherwise. Let G be a nonempty open set in R. Choose $x \in G$ such that $x \neq 0$. Then $f(x) > \sup g^{-1}(V)$. Thus $g(f(x)) \notin V$. This shows that $g(f)$ is not quasicontinuous at the point 0.

References

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