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A CHARACTERIZATION OF NON-ATOMIC PROBABILITIES ON [0,1] WITH NOWHERE DENSE SUPPORTS

For a countably additive Borel probability measure μ on $[0,1]$, let $\{T_i(\mu) : i \in N\}$ be an enumeration of the connected components of $[0,1]\backslash\text{supp}(\mu)$. These are the intervals of constancy of the cumulative distribution function F_μ. For all i let $y_i(\mu)$ be the value of F_μ on $T_i(\mu)$.

Proposition 1 *μ is non-atomic with supp(μ) nowhere dense iff $\{y_i(\mu) : i \in N\}$ is dense in $[0,1]$.*

Proof: Suppose that μ is non-atomic with nowhere dense support. Since μ is non-atomic F_μ is continuous and $\{F_\mu(x) : x \in \text{supp}(\mu)\} = [0,1]$. If $0 \leq y_1 < y_2 \leq 1$ are $F_\mu(x_1)$ and $F_\mu(x_2)$ with $x_1 < x_2$ in supp(μ) there is an interval $T_i(\mu)$ between x_1 and x_2 since supp(μ) is nowhere dense. Thus $y_1 < y_i(\mu) < y_2$. This establishes density of $\{y_i(\mu) : i \in N\}$ in $[0,1]$.

Assume density of $\{y_i(\mu) : i \in N\}$. μ must be non-atomic for otherwise there would be an $x \in [0,1]$ so that $F_\mu(x^-) = \lim_{z \uparrow x} F_\mu(z) < F_\mu(x)$. In this case no $y_i(\mu)$ would be in $(F_\mu(x^-), F_\mu(x))$ contradicting density. supp(μ) must be nowhere dense for if $\phi \neq (x_1, x_2) \subset \text{supp}(\mu)$ then $F_\mu(x_1) < F_\mu(x_2)$ so $y_i(\mu) \in (F_\mu(x_1), F_\mu(x_2))$ for some $i \in N$ hence $T_i(\mu)$ is in (x_1, x_2) which is impossible since $T_i(u) \cap \text{supp}(u) = \phi$. Thus supp($\mu$) is nowhere dense.

□

The intervals $\{T_i(\mu) : i \in N\}$ are non-overlapping and are ordered by $T_i(\mu) < T_j(\mu)$ iff $x_i \in T_i(\mu)$ and x_j in $T_j(\mu)$ implies $x_i < x_j$. The mapping $y_i \rightarrow T_i(\mu)$ is an order isomorphism. $\{y_i(\mu) : i \in N\}$ has maximum 1 (minimum 0) iff $\{T_i(\mu) : i \in N\}$ has a maximum containing 1 (minimum containing 0) iff $1 \notin \text{supp}(\mu)$ ($0 \notin \text{supp}(\mu)$). Allowing for different possible order types the converse is true. If K is a perfect nowhere dense subset of $[0,1]$ and the countable dense subset $\{y_i : i \in N\}$ of $[0,1]$ has extrema of the same type as the components $\{T_i : i \in N\}$ of $[0,1]\backslash K$ there is an order isomorphism $T_i \leftrightarrow y_i$ (see Theorem 1 page 160 of Fraenkel [1961]). For such an isomorphism define $F(x) = y_i$ if $x \in T_i$ to obtain a non-decreasing function from $[0,1]\backslash N \rightarrow [0,1]$ which has a right continuous extension (which is continuous

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by density of $\{y_i : i \in N\}$) to a surjection $F : [0,1] \to$ which is F_μ for some Borel probability μ. This yields this proposition.

Proposition 2 *Let K be a perfect nowhere dense subset of $[0,1]$ and let $\{y_i : i \in N\} = Y$ be a dense subset of $[0,1]$ with the same order type as the components of $[0,1]\backslash K$. There is a Borel probability μ with supp(μ) $= K$ and $\{y_i\} = \{y_i(\mu)\}$ for all i.*

It should be remarked that μ is uniquely determined by specification of a particular order isomorphism between Y and components of $[0,1]\backslash K$. The possible μ are in 1-1 correspondence with the order automorphisms of Y.

Proposition 3 *If $\mathcal{F}$ is the algebra of m-measurable sets for m a non-atomic probability measure with support $[0,1]$, $\{N_n : n \in N\}$ is a sequence of perfect nowhere dense sets in $[0,1]$ with $1 = \lim_{n\to\infty} m(N_n)$ and m_n is the restriction of m to N_n normalized to be a probability then $\lim_{n\to\infty} m_n(F) = m(F)$ if $F \in \mathcal{F}$.*

Proof: Let $\{N_n\}$ be a sequence of perfect nowhere dense sets with $1 = \lim_{n\to\infty} m(N_n)$. For each n, m_n is defined, for $F \in \mathcal{F}$, by $m_n(F) = m(F \cap N_n)/m(N_n)$. Since $m([0,1]\backslash N_n) \to 0$ as $n \to \infty$. It is immediate from this that $m_n(F) \to m(F)$ for any $F \in \mathcal{F}$.

□

Proposition 3, as is seen from the proof, is valid in great generality. $\mathcal{F}$ need only be an algebra, m only finitely additive and $\{N_n : n \in N\}$ a sequence in $\mathcal{F}$ with $\lim_{n\to\infty} m(N_m) - 1$. Cannizzo however singles out the perfect nowhere dense sets in $[0,1]$ since for countably additive non-atomic Baire probability measures here or in any Polish or compact Hansdorff space such a sequence of perfect nowhere dense sets always exists.

Propositions 1 is not valid in the absence of countable additivity. One must interpret supp(μ) for a finitely additive Borel measure as the intersection of all closed sets of full measure but supp(μ) may fail to be of full measure with $\mu(\text{supp}(\mu)) = 0$ a possibility. The cumulative distribution function $\mathcal{F}_\mu$ is defined as usual but may fail to be continuous from the left. If F_μ is continuous then μ is non-atomic but it may be the case that μ is non-atomic yet have F_μ failing left or right continuity. This Lemma gives some indication of arbitrariness of $\mathcal{F}_\mu$ when countable additivity is not required. This is extended in Proposition 5.

Lemma 5 *If x is in $[0,1]$ there is a non-atomic finitely additive Borel probability measure μ on $[0,1]$ with F_μ the indicator function for $(x,1]$ if $x < 1$ or for $[x,1]$ if $x > 0$.*

Proof: First assume that $x = 0$. It must be shown that there is a non-atomic Borel probability μ with $\mu(\{0\}) = 0$ and with $\mu([0,\epsilon]) = 1$ if $\epsilon > 0$. It is easily seen that if $(x_n : n \in N)$ is a strictly decreasing sequence in $[0,1]$ with $\lim_{n\to\infty} x_n = 0$ then any finitely additive non-atomic probability ν o 2^N induces a non-atomic probability μ on 2^A where $A = \{x_n : n \in N\}$ under the map $n \to x_n$. Extend μ to a finitely additive Borel probability measure on $[0,1]$ with $\mu([0,1]\backslash A) = 0$. It is immediate that $\mu\{0\}) = 0$ and that $\mu([0,x_n]) = 1$ since $\mu(\{x_m : m \geq n\}) = 1$ for any $n \in N$. Since $x_n \to 0$ the result follows.

For general $x < 1$ in the preceding argument one should use a strictly decreasing sequence (x_n) with $x = \lim_{n\to\infty} x_n$ to obtain μ with $F_\mu = I_{(x,1]}$. A similar construction works to give μ with $\mathcal{F}_\mu = I_{[x,1]}$ if $x > 0$.

□

deFinetti [1972] realized that a non-atomic finitely additive measure μ could have $\mu((x-\varepsilon, x+\varepsilon)) \geq \lambda > 0$ for all $\varepsilon > 0$. In general when such an x exists μ was called *agglutinated*. Agglutination is equivalent to the presence of a jump in F_μ so μ is non-agglutinated iff F_μ is continuous. It is easily seen that positive linear combinations of measures as in Proposition 4 yield measures μ so that the entire variation of F_μ is taken up in jumps and that any increasing F on $[0,1]$ with $F(0) \geq 0$ and $F(1) = 1$ whose jumps sum to 1 is F_μ for μ a countable convex combination of measures in Proposition 4. Such μ could be called *totally agglutinated* yet may be non-atomic. As a result of the following proposition both Propositions 1 and 2 retain their validity if non-atomic countably additive measures are replaced by non-agglutinated finitely additive measures.

Proposition 5 *If F is an increasing function on $[0,1]$ with $F(0) \geq 0$ and $F(1) = 1$ there is a finitely additive non-atomic Borel measure μ on $[0,1]$ with $F_\mu = F$ which gives probability 1 to the rationals.*

Proof: In Proposition 4, as may be seen by the proof one may find for any x non-atomic probabilities μ_x and μ_x^+ with $F_{\mu_x} = I_{(x,1]}$ and $F_{\mu_x^+} = I_{[x,1]}$ which give measure 1 to the rationals (basing the proof of Proposition 4 on sequences of rationals). As a result, if the jumps of F equal 1 a μ exists with $F = F_\mu$ and with μ a countable convex combination of such μ_x, μ_x^+. To establish the proposition it is only necessary to consider the case with F continuous. In this case for any rational $r \in [0,1]$ let x_r be such that $F(x_r) = r$. F is the uniform limit of $\{F_n : n \in N\}$ where $F_n = \sum_{k=0}^{n} \frac{1}{n} I_{[x_{k/n}]}$. Each F_n is F_{μ_n} where μ_n is a Borel probability giving measure 1 to the rationals. Let δ be a $\{0,1\}$-valued probability measure on 2^N annihilating singletons. For A Borel let $\mu(A) = \int_N \mu_n(A)\delta(dn)$. If $\mathcal{U}$ is the free ultrafilter on N corresponding to δ then $\mu(A) = \lim_{n\in\mathcal{U}} \mu_n(A)$. As a result if $t \in [0,1]$ then $F_\mu(t) = \mu([0,t]) = \lim_{n\in\mathcal{U}} F_{\mu_n}(t) = \lim_{n\to\infty} F_{\mu_n}(t) = F(t)$. Since $\mu_n(Q \cap [0,1]) = 1$ for all n we have $\mu(Q \cap [0,1]) = 1$. Since F_μ is continuous μ is non-atomic.

□

References

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