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ON THE DANIÉLL INTEGRAL

H. M. MacNeille in [2] suggested the following simple definition of Lebesgue integrable functions:

A real function f (defined on $\mathbb{R}^N$) is Lebesgue integrable if there exists a sequence of step functions $f_1, f_2, f_3, \ldots$ such that

(*) $\sum_{n=1}^{\infty} \int |f_n| < \infty$,

(**) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *for every* $x \in \mathbb{R}^N$ *for which* $\sum_{n=1}^{\infty} |f_n(x)| < \infty$.

Systematic use of the above approach in [3] shows that it gives a very fast and natural way of developing the theory of Lebesgue integral as well as the Bochner integral. In this note we show that the same method can be successfully used to construct the Daniell integral. In the standard approach one has to introduce auxiliary spaces of the so-called "over-functions" and "under-functions" which are used only as a step in the construction and are not needed later. In the present method the integral and the space of integrable functions are introduced in one step without any additional constructions. Moreover, our approach simplifies proofs of important theorems on the integral.

Some proofs in this note are adaptations of proofs of corresponding properties of the Lebesgue or Bochner integral in [3]. Other proofs are actually simpler in the abstract setting of the Daniell integral. Since the aim of this note is to show that the described approach simplifies the construction of the Daniell integral no proofs are omitted.

It is interesting that although the Daniell integral is discussed in [3] the traditional approach is used.

A triple $(K, \mathfrak{U}, \int)$ will be called a *Daniell space* if K is a nonempty set, $\mathfrak{U}$ is a Riesz space (i.e., vector lattice) of real valued functions on K, and $\int$ is a real linear functional on $\mathfrak{U}$ such that

(1) $\int f \geq 0$ whenever $f \geq 0$,

(2) $\int f_n \to 0$ for every non-increasing sequence of functions $f_n \in \mathfrak{U}$ such that $f_n(x) \to 0$ for every $x \in K$.

For the lattice operations we use the following notation: $f \cup g = max(f, g)$ and $f \cap g = min(f, g)$.

Definition 1. Let f be a real function on K. If there exist functions $f_n \in \mathfrak{U}$, $n = 1, 2, 3, \ldots$, such that

(a) $\sum_{n=1}^{\infty} \int |f_n| < \infty$,

(b) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in K$ for which $\sum_{n=1}^{\infty} |f_n(x)| < \infty$,

then we write

$$f \simeq_{\mathfrak{U}} \sum_{n=1}^{\infty} f_n \quad \text{or} \quad f \simeq_{\mathfrak{U}} f_1 + f_2 + f_3 + \ldots$$

A Daniell space $(K, \mathfrak{U}, \int)$ will be called *complete* if $f \simeq_{\mathfrak{U}} \sum_{n=1}^{\infty} f_n$ implies that $f \in \mathfrak{U}$.

The symbol "$\simeq_{\mathfrak{U}}$" will be abbreviated to "$\simeq$" whenever the Riesz space $\mathfrak{U}$ is unambiguous.

Definition 2. Given a Daniell space $(K, \mathfrak{U}, \int)$, denote by $\mathfrak{U}^*$ the space of all real valued functions f on K for which there exists a sequence of functions $f_1, f_2, f_3, \ldots \in \mathfrak{U}$ such that $f \simeq \sum_{n=1}^{\infty} f_n$. For $f \in \mathfrak{U}^*$ define

$$\int^* f = \sum_{n=1}^{\infty} \int f_n.$$

It is necessary to prove that the value $\int^* f$ is independent of a particular sequence $\{f_n\}$. This easily follows from Theorem 4. In the proof of that theorem we use the following known lemma.

Lemma 3. *If the sequences $\{g_n\}$ and $\{h_n\}$, $g_n, h_n \in \mathfrak{U}$, are non-decreasing and $\lim_{n \to \infty} h_n(x) \leq \lim_{n \to \infty} g_n(x)$ for every $x \in K$, then*

$$\lim_{n \to \infty} \int h_n \leq \lim_{n \to \infty} \int g_n.$$

Proof. Fix $k \in \mathbb{N}$. Since the functions $h_k - (h_k \cap g_n)$ $(n = 1, 2, \ldots)$ form a non-increasing sequence which converges to zero at every point of K, we have

$$\lim_{n \to \infty} \left(\int h_k - \int (h_k \cap g_n) \right) = 0$$

and hence

$$\int h_k = \lim_{n \to \infty} \int (h_k \cap g_n) \leq \lim_{n \to \infty} \int g_n.$$

By letting $k \to \infty$ we obtain the desired inequality.

Theorem 4. *If* $f \simeq \sum_{n=1}^{\infty} f_n$ *and* $f \geq 0$, *then* $\sum_{n=1}^{\infty} \int f_n \geq 0$.

Proof. First note that (a) ensures the convergence of $\sum_{n=1}^{\infty} \int f_n$. To show that $\sum_{n=1}^{\infty} \int f_n \geq 0$, fix $p \in \mathbf{N}$. Then, for $n \in \mathbf{N}$, define

$$g_n = f_1 + \ldots + f_p + |f_{p+1}| + \ldots + |f_{p+n}|$$

and

$$h_n = g_n \cup 0.$$

The sequences $\{g_n\}$ and $\{h_n\}$ are non-decreasing, $g_n, h_n \in \mathfrak{U}$ and $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n$ (possibly $+\infty$). The equality of the limits follows from $f \geq 0$ and (b). Therefore, $\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int h_n \geq 0$, by Lemma 3. Thus

$$\int f_1 + \ldots + \int f_p + \int |f_{p+1}| + \int |f_{p+2}| + \ldots \geq 0.$$

Hence, by letting $p \to \infty$, we obtain $\sum_{n=1}^{\infty} \int f_n \geq 0$.

Corollary 5. *If* $f \simeq \sum_{n=1}^{\infty} f_n$ *and* $f \simeq \sum_{n=1}^{\infty} g_n$, *then* $\sum_{n=1}^{\infty} \int f_n = \sum_{n=1}^{\infty} \int g_n$.

Proof. Since $0 \simeq f_1 - g_1 + f_2 - g_2 + \ldots$, we have

$$\sum_{n=1}^{\infty} \int f_n - \sum_{n=1}^{\infty} \int g_n \geq 0.$$

Similarly

$$\sum_{n=1}^{\infty} \int g_n - \sum_{n=1}^{\infty} \int f_n \geq 0,$$

and thus

$$\sum_{n=1}^{\infty} \int g_n = \sum_{n=1}^{\infty} \int f_n.$$

Corollary 6. *If* $f \in \mathfrak{U}$, *then* $f \in \mathfrak{U}^*$ *and* $\int^* f = \int f$.

Proof. If $f \in \mathfrak{U}$, then $f \simeq f + 0 + 0 + \ldots$

Corollary 7. $\mathfrak{U}^*$ *is a vector space and* $\int^*$ *is a linear functional on* $\mathfrak{U}^*$. *Moreover, if* $f, g \in \mathfrak{U}^*$ *and* $f \leq g$, *then* $\int^* f \leq \int^* g$.

Proof. If $f \simeq \sum_{n=1}^{\infty} f_n$, $g \simeq \sum_{n=1}^{\infty} g_n$ and $\lambda \in \mathbf{R}$, then

$$f + g \simeq f_1 + g_1 + f_2 + g_2 + \ldots$$

and

$$\lambda f \simeq \lambda f_1 + \lambda f_2 + \ldots$$

Consequently

$$\int^*(f+g) = \int^* f + \int^* g$$

and

$$\int^*(\lambda f) = \lambda \int^* f.$$

If $f, g \in \mathfrak{U}^*$ and $f \leq g$, then $g - f \in \mathfrak{U}^*$ and $g - f \geq 0$. Hence $\int^*(g-f) \geq 0$, by Theorem 4, and thus

$$\int^* f \leq \int^* g.$$

Theorem 8. *If* $f \in \mathfrak{U}^*$, *then* $|f| \in \mathfrak{U}^*$ *and* $\left|\int^* f\right| \leq \int^* |f|$. *Moreover, if* $f \simeq \sum_{n=1}^{\infty} f_n$, *then*

$$\int^* |f| = \lim_{n\to\infty} \int |f_1 + \ldots + f_n|.$$

Proof. Let $f \simeq \sum_{n=1}^{\infty} f_n$. Define

$$Z = \left\{x \in K : \sum_{n=1}^{\infty} |f_n(x)| < \infty\right\}$$

and

$$s_n = f_1 + \ldots + f_n, \quad n = 1, 2, 3, \ldots$$

Then $f(x) = \lim_{n\to\infty} s_n(x)$ for every $x \in Z$. Hence also $|f(x)| = \lim_{n\to\infty} |s_n(x)|$ for $x \in Z$, i.e.,

$$|f(x)| = |s_1(x)| + (|s_2(x)| - |s_1(x)|) + (|s_3(x)| - |s_2(x)|) + \ldots \quad \text{for } x \in Z.$$

Put $g_1 = |s_1|$ and $g_n = |s_n| - |s_{n-1}|$ for $n \geq 2$. We will show that

$$|f| \simeq g_1 + f_1 - f_1 + g_2 + f_2 - f_2 + \ldots$$

To check (a) it suffices to note that for $n \geq 2$ we have

$$|g_n| = \big||s_n| - |s_{n-1}|\big| \leq |s_n - s_{n-1}| = |f_n|,$$

and hence

$$\int |g_n| \leq \int |f_n|.$$

To verify (b) observe that

$$|f(x)| = \sum_{n=1}^{\infty} g_n(x) \quad \text{for } x \in Z,$$

and that the series

$$g_1(x) + f_1(x) - f_1(x) + g_2(x) + f_2(x) - f_2(x) + \ldots$$

is not absolutely convergent if $x \notin Z$. Thus $|f| \in \mathfrak{U}^*$.

Since $f \leq |f|$ and $-f \leq |f|$, we have $\int^* f \leq \int^* |f|$ and $-\int^* f \leq \int^* |f|$, by Corollary 7, and hence $\left|\int^* f\right| \leq \int^* |f|$.

Finally,

$$\int^{*}|f| = \sum_{n=1}^{\infty}\int g_n = \lim_{n\to\infty}\int|s_n| = \lim_{n\to\infty}\int|f_1+\ldots+f_n|.$$

Corollary 9. *$\mathfrak{U}^{*}$ is closed under the lattice operations.*

Proof. The assertion follows from

$$f\cup g=\tfrac{1}{2}(f+g+|f-g|),\quad f\cap g=\tfrac{1}{2}(f+g-|f-g|),$$

by Corollary 7 and Theorem 8.

Lemma 10. *If $f\in\mathfrak{U}^{*}$, then for every $\epsilon>0$ there exists a sequence of functions $f_1, f_2, \ldots \in\mathfrak{U}$ such that $f\simeq\sum_{n=1}^{\infty}f_n$ and*

$$\sum_{n=1}^{\infty}\int|f_n| \le \int^{*}|f|+\epsilon.$$

Proof. Let $f\simeq\sum_{n=1}^{\infty}g_n$ and $\sum_{n_1+1}^{\infty}\int|g_n|<\frac{1}{2}\epsilon$. Since, by Theorem 8, $\int^{*}|f|=\lim_{n\to\infty}\int|g_1+\ldots+g_n|$, there exists $n_2\in\mathbf{N}$ such that

$$\int|g_1+\ldots+g_n| < \int^{*}|f|+\frac{\epsilon}{2}$$

for every $n\ge n_2$. Let $n_0=max\{n_1,n_2\}$. Define $f_1=g_1+\ldots+g_{n_0}$ and $f_n=g_{n_0+n-1}$ for $n\ge 2$. Then

$$f\simeq\sum_{n=1}^{\infty}f_n$$

and

$$\sum_{n=1}^{\infty}\int|f_n| = \int|g_1+\ldots+g_{n_0}|+\sum_{n_0+1}^{\infty}\int|g_n| \le \int^{*}|f|+\frac{\epsilon}{2}+\frac{\epsilon}{2},$$

completing the proof.

Theorem 11. *If $f\simeq\sum_{n=1}^{\infty}f_n$ $(f_n\in\mathfrak{U}^{*})$, then $f\in\mathfrak{U}^{*}$ and $\int^{*}f=\sum_{n=1}^{\infty}\int^{*}f_n$.*

Proof. Let $f\simeq\sum_{n=1}^{\infty}f_n$ $(f_n\in\mathfrak{U}^{*})$. Choose $g_{in}\in\mathfrak{U}$, $i,n\in\mathbf{N}$, such that

$$f_i\simeq\sum_{n=1}^{\infty}g_{in}$$

and

$$\sum_{n=1}^{\infty}\int|g_{in}| \le \int^{*}|f_i|+2^{-i}\quad\text{for } i=1,2,\ldots$$

Let $\{h_n\}$ be a sequence arranged from all the functions g_{in}. Then clearly

$$f\simeq\sum_{n=1}^{\infty}h_n$$

which implies $f\in\mathfrak{U}^{*}$ and $\int^{*}f=\sum_{n=1}^{\infty}\int^{*}f_n$.

Corollary 12. *For every non-increasing sequence of functions* $f_n \in \mathfrak{U}^*$ *such that* $f_n(x) \to 0$ *for every* $x \in K$ *we have* $\int^* f_n \to 0$.

Proof. It suffices to observe that

$$0 \simeq f_1 + (f_2 - f_1) + (f_3 - f_2) + \ldots$$

and use Theorem 11.

Theorem 4, Corollary 7, Corollary 9, and Corollary 12 show that $(K, \mathfrak{U}^*, \int^*)$ is a Daniell space. $(K, \mathfrak{U}^*, \int^*)$ is a complete Daniell space, by Theorem 11. The above results constitute a proof of the following theorem.

Theorem 13. *Given a Daniell space* $(K, \mathfrak{U}, \int)$ *there exists a smallest complete Daniell space* $(K, \mathfrak{U}^*, \int^*)$ *such that* $\mathfrak{U} \subset \mathfrak{U}^*$ *and* $\int^* f = \int f$ *for all* $f \in \mathfrak{U}$.

Remark. In [4] the method of absolutely convergent series is used in the construction of the complete measure from a measure on an algebra of sets.

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