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SPECTRAL RADIUS OF NONSINGULAR TRANSFORMATIONS

We say that T is an invertible, nonsingular, ergodic transformation on a probability space $(X, \mathcal{B}, \mu)$ if $T:X \to X$ is one to one, $T(A)$ and $T^{-1}(A) \in \mathcal{B}$ whenever $A \in \mathcal{B}$, $\mu(T(A)) > 0$ and $\mu(T^{-1}(A)) > 0$ whenever $\mu(A) > 0$, and $\mu(A) = 0$ or 1 whenever $T(A) = A$. Flytzanis [1] introduced the spectral radius as an invariant for such transformations as follows: For every $A \in \mathcal{B}$ with $\mu(A) > 0$, let $r(T,A)$ denote the radius of convergence of the power series $\sum_{n=0}^{\infty} \mu(\Delta A_n)\, x^n$ where $A_n = \bigcup_{j=-n}^{n} T^j A$, $A_0 = A$, $A_{-1} = \varnothing$, and $\Delta A_n = A_n - A_{n-1}$. The spectral radius $r(T)$ is then equal to $\inf\{ r(T,A) : \mu(A) > 0\}$. It is clear that $r(T) \geq 1$, and if T is a periodic transformation (T^p = identity for some $p \geq 1$), then $r(T) = \infty$. We will assume that μ is nonatomic, so that, since T is ergodic, it can not be periodic. The purpose of this note is to prove:

Theorem. Let T be any invertible nonsingular ergodic transformation acting on a nonatomic probability space. Then $r(T) = 1$.

Robertson [3] showed that the following property implies that $r(T) = 1$.

Property 1. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all positive integers k there exists a set A (depending on ε, δ, and k) such that $0 < \mu(A) < \varepsilon$ and $\mu(\bigcap_{j=-k}^{k} T^j A) > \delta$.

For the sake of completeness we include a proof of this implication.

Lemma 1. Assume that μ *is nonatomic,* $\mu(X) = 1$, *and that* T *is ergodic, invertible and nonsingular. Then* $r(T,A)$ *is less than or equal to the radius of convergence of the power series* $\sum_{n=0}^{\infty} \mu(A_n') x^n$, *where* $A_n' = X - A_n$.

Proof. Let $\mu(A) > 0$ and let x_0 be such that $0 < x_0 < r(T,A)$, but $x_0 \neq 1$. Since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{j=-\infty}^{\infty} T^j A$, is an invariant set of positive measure, it has measure one, and thus, except for a set of measure zero, $A_n' = \bigcup_{j=n+1}^{\infty} \Delta A_j$.

Hence

$$\sum_{n=0}^{\infty} \mu(A_n')\, x_0^n = \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \mu(\Delta A_j)\, x_0^n$$

$$= \sum_{j=1}^{\infty} \mu(\Delta A_j) \left(\sum_{n=0}^{j-1} x_0^n \right)$$

$$= \sum_{j=1}^{\infty} (1 - x_0^j)/(1 - x_0)\, \mu(\Delta A_j)$$

$$= \left[\sum_{j=1}^{\infty} \mu(\Delta A_j) - \sum_{j=1}^{\infty} \mu(\Delta A_j)\, x_0^j \right] / [1 - x_0]$$

$$< \infty .$$

Thus the radius of convergence of the power series $\sum_{n=0}^{\infty} \mu(A_n')\, x^n$ is greater than or equal to x_0. Letting x_0 approach $r(T,A)$ we have the desired result.

Corollary 2. $r(T,A) \leq [\limsup_{n \to \infty} (\mu(A_n'))^{1/n}]^{-1}$.

Proof. This follows from the standard formula for the radius of convergence.

Lemma 3. *Suppose property* 1 *is satisfied, i.e. for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every positive integer* k, *there is a set* $A \in \mathcal{B}$ (*depending on* ε, δ, *and* k) *such that* $\mu(A) < \varepsilon$ *and* $\mu(\bigcap_{j=-k}^{k} T^j A) > \delta$. *Then there exists a set* $B \in \mathcal{B}$ *such that* $r(T,B) = 1$.

Proof. Let $\varepsilon_n > 0$ be such that $\sum_{n=0}^{\infty} \varepsilon_n < 1$. Let δ_n correspond to ε_n in the hypothesis of the lemma. Let $r_n < 1$ be such that $\lim_{n\to\infty} r_n = 1$. Choose k_n such that $r_n{}^{k_n} < \delta_n$. Finally let A_n be the set corresponding to ε_n, δ_n and k_n in the hypothesis of the lemma. Set $B = \bigcap_{n=0}^{\infty} A(n)'$. Then $\mu(B') = \mu(\bigcup_{n=0}^{\infty} A(n)) < \sum_{n=0}^{\infty} \varepsilon_n < 1$. Thus $\mu(B) > 0$. Further

$$
\begin{aligned}
r(T,B) &\le [\limsup_{n\to\infty} \mu(\bigcap_{j=-n}^{n} T^j(B'))^{1/n}]^{-1} \\
&\le [\limsup_{n\to\infty} \mu(\bigcap_{j=-k_n}^{k_n} T^j(B'))^{1/k_n}]^{-1} \\
&\le [\limsup_{n\to\infty} \mu(\bigcap_{j=-k_n}^{k_n} T^j A(n))^{1/k_n}]^{-1} \\
&\le (\limsup_{n\to\infty} \delta_n{}^{1/k_n})^{-1} \le (\limsup_{n\to\infty} r_n)^{-1} = 1 .
\end{aligned}
$$

Since $r(T,B)$ is always greater than or equal to one, we see that $r(T,B) = 1$. This proves the lemma.

It was shown by Robertson [3] that every measure preserving transformation satisfies property 1. Here we prove the same for invertible nonsingular ergodic transformations.

Theorem. _Let_ T _be an invertible nonsingular ergodic transformation acting on a nonatomic probability space_ $(X, \mathcal{B}, \mu)$. _Let_ ε _and_ δ _be any two numbers such that_ $0 < \delta < \varepsilon < 1$. _Then for every positive integer_ k _there exists a set_ $A \in \mathcal{B}$ _such that_ $\mu(A) < \varepsilon$ _and_ $\mu(\bigcap_{j=-k}^{k} T^j A) > \delta$. _In particular_ $r(T) = 1$.

Proof. Choose α such that $\delta < \alpha < \varepsilon$. For $A \in \mathcal{B}$, write $\nu_k(A) = \sum_{j=-k}^{k} \mu(T^j A)$. Then ν_k is absolutely continuous with respect to μ. Hence there exists an $\eta > 0$ such that $\mu(A) < \eta$ implies that $\sum_{j=-k}^{k} \mu(T^j A) < (\alpha - \delta)/4$ (η will depend on k). Let N be an integer larger than $k + 1/(2\eta)$. Then choose $0 < \eta' < \eta$ such that $\mu(A) < \eta'$ implies that $\sum_{j=-2N}^{2N} \mu(T^j A) < (\alpha - \delta)/4$. We next apply Rohlin's theorem for nonsingular transformations which can be found for example in Friedman

[2] (Lemma 7.9). There exists a set $F \in \mathcal{B}$ such that

$F, TF, \dots, T^{2N-1}F$ are disjoint and $\mu(R) < \eta'$ where R is the complement of $\bigcup_{j=1}^{2N} T^{j-1}F$. There is some i such that $k \le i < 2N - k$ and $\mu(T^iF) < 1/(2(N-k)) < \eta$. Therefore $\sum_{j=-k}^{k} \mu(T^{j+i}F) < (\alpha - \delta)/4$. Since $\mu(R) < \eta'$, we have $\sum_{j=-k}^{-1} \mu(T^{j+i}R) < (\alpha - \delta)/4$. Using the fact that $T^{2N}F \subseteq F \cup R$ we have the following:

$$\sum_{j=2N-k}^{2N-1} \mu(T^{j+i}F) = \sum_{j=-k}^{-1} \mu(T^{j+i}T^{2N}F)$$

$$\le \sum_{j=-k}^{-1} \mu(T^{j+i}(F \cup R))$$

$$\le \sum_{j=-k}^{-1} \mu(T^{j+i}F) + \sum_{j=-k}^{-1} \mu(T^{j+i}R)$$

$$< (\alpha - \delta)/2 .$$

Finally, since $\sum_{j=0}^{2N-1} \mu(T^j(T^i(F))) = \mu(T^i(\bigcup_{j=0}^{2N-1} T^j(F))) = 1 - \mu(T^i(R)) \ge 1 - \sum_{j=-2N}^{2N} \mu(T^j(R)) \ge 1 - (\alpha - \delta)/4$ and μ is nonatomic, we may choose $B \subseteq T^iF$ such that $\varepsilon - (\alpha - \delta)/4 < \sum_{j=0}^{2N-1} \mu(T^jB) < \varepsilon$. Now

$B, TB, \dots, T^{2N-1}B$ are disjoint and $\sum_{j=0}^{k-1} \mu(T^jB) < (\alpha - \delta)/4$, and

$\sum_{j=2N-k}^{2N-1} \mu(T^jB) < (\alpha - \delta)/2$. Set $A = \bigcup_{j=0}^{2N-1} T^jB$. Then for $-k \le j \le k$ we have $T^jA \supseteq \bigcup_{p=k}^{2N-k-1} T^pB$. Hence $\bigcap_{j=-k}^{k} T^jA \supseteq \bigcup_{p=k}^{2N-k-1} T^pB$. Thus

$$\mu\left(\bigcap_{j=-k}^{k} T^jA\right) > \sum_{j=0}^{2N-1} \mu(T^jB) - \sum_{j=0}^{k-1} \mu(T^jB) - \sum_{j=2N-k}^{2N-1} \mu(T^jB)$$

$$> \varepsilon - (\alpha - \delta) = \delta + \varepsilon - \alpha$$

$$> \delta .$$

This completes the proof of the theorem.

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