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F.S. Cater, Department of Mathematics, Portland State University, Portland, Oregon 97207

Some Higher Dimensional Marcinkiewicz Theorems

Let I denote the compact interval $[0,1]$ and let $C(I)$ denote the Banach space of continuous real valued functions on I under the sup norm, $\| f \| = \sup_{x\in I} | f(x) |$. In [1], J. Marcinkiewicz proved that there exists an $f \in C(I)$ such that for each measurable function g on I there is a sequence of positive numbers (h_j) converging to 0 and depending only on g for which $\lim_{j\to\infty}(f(x+h_j)-f(x))/h_j = g(x)$ almost everywhere on I. What is striking is that the same function f suffices for all measurable functions g. Marcinkiewicz also proved that in the sense of Baire category, most functions in $C(I)$ can be chosen for f.

In this note, for each $n > 0$ we define a formal linear combination, $F_n(x,h)$, of functions of the form $f(x+jh)$ $(j = 0, \pm1, \pm2, \pm3, \ldots)$ over the integers. We will find an $f \in C(I)$ such that for each measurable function g on I and each n, there is a sequence (h_k) of positive numbers converging to 0 and depending only on g and n for which $\lim_{k\to\infty} F_n(x,h_k)/h_k^n = g(x)$ almost everywhere on I. Moreover, in the sense of Baire category most functions in $C(I)$ can be chosen for f.

Let f denote any real valued function. Put $F_1(x,h) = f(x+h) - f(x-h)$ and $F_2(x,h) = f(x+h) + f(x-h) - 2f(x)$. By induction for $n \geq 3$, we put $F_n(x,h) = 2^{n-2}F_{n-2}(x,h) - F_{n-2}(x,2h)$. Then $F_n(x,h)$ is defined for all positive integers n, and $F_n(x,h)$ is the type of formal linear combination described in the preceding paragraph. For example, $F_3(x,h) = 2f(x+h) - 2f(x-h) - f(x+2h) + f(x-2h)$ and $F_4(x,h) = 2^2f(x+h) + 2^2f(x-h) - 2^3f(x) - f(x+2h) - f(x-2h) + 2f(x)$.

We need some limits involving F_n when f is a polynomial function of x.

Lemma 1. Let $p(x)$ be a polynomial function of x and let $P_n(x,h)$ be the function formed from p in the same way as $F_n(x,h)$ was formed from f (above). Then for each positive integer n there is a nonzero constant c_n, independent of p, such that $\lim_{h\to 0} P_n(x,h)/h^n = c_n p^{(n)}(x)$.

Proof. By the Taylor expansion,

$$p(x+h) = p(x) + p'(x)h + p''(x)h^2/2! + p^{(3)}(x)h^3/3! + p^{(4)}(x)h^4/4! + \ldots.$$

By direct computation we obtain for $n \geq 3$

$$P_n(x,h) = c_n p^{(n)}(x) h^n + h^{n+2}(\ldots)$$

where the factor $(\ldots)$ is a linear combination of terms of the form $p^{(j)}(x)$ over polynomials in h, and where

$$\begin{aligned} c_{2k} &= 2(2^2 - 2^{2k})(2^4 - 2^{2k})\ldots(2^{2k-2} - 2^{2k})/(2k)! \quad \text{and} \\ c_{2k+1} &= 2(2 - 2^{2k+1})(2^3 - 2^{2k+1})\ldots(2^{2k-1} - 2^{2k+1})/(2k+1)!. \end{aligned}$$

To see this use induction on k for P_{2k} and then P_{2k+1}. The result follows from this. □

Clearly Lemma 1 will work for some functions more general than polynomials in x, but we require it only for polynomials. We turn now to some ad hoc definitions and notation.

Definition. We say that a function $f \in C(I)$ is nearly constant on I if almost every $x \in I$ lies in an open interval on which f is constant. We say that $f \in C(I)$ is nearly polynomial on I if almost every $x \in I$ lies in an open interval on which f coincides with a polynomial in x.

Thus any nearly constant function on I must be nearly polynomial on I. A primitive of a nearly polynomial function is nearly polynomial. And for $\varepsilon > 0$ and continuous g, there is a nearly constant f such that $\| f - g \| < \varepsilon$.

Let $C_n(I)$ denote the set of functions in $C(I)$ that have continuous n-th derivatives everywhere on I. Then $C_n(I)$ is a Banach space under the norm $\ll f \gg_n = \sum_{i=0}^{n} \| f^{(i)} \|$. Here $f^{(0)}$ means f. To be consistent, we put $C_0(I) = C(I)$ and $\ll f \gg_0 = \| f \|$.

Lemma 2. Let $g_0 \in C(I)$, $g_1 \in C_{n-1}(I)$, $g_2 \in C(I)$ for some $n \geq 1$. Let $\varepsilon > 0$. Then

(i) there is a nearly polynomial function $f_1 \in C_{n-1}(I)$ such that $\ll f_1 - g_1 \gg_{n-1} < \varepsilon$ and $| f_1^{(n)}(x) - g_0(x) | < \varepsilon$ almost everywhere on I.

(ii) there is a nearly polynomial function $f_2 \in C_{n-1}(I)$ such that $\| f_2 - g_2 \| < \varepsilon$ and $| f_2^{(n)}(x) - g_0(x) | < \varepsilon$ almost everywhere on I.

Proof (i). Use the Weierstrass Approximation Theorem to select a polynomial function $p_1(x)$ such that

$$(*) \qquad \| p_1^{(n)} - g_0 \| < \varepsilon.$$

Because $g_1^{(n-1)} - p_1^{(n-1)}$ is continuous, there is a nearly constant continuous function q_1 such that $\| q_1 - g_1^{(n-1)} + p_1^{(n-1)} \| < \varepsilon/n$.

$$\begin{aligned}
\text{For } x \in I, \text{let } q_2(x) &= g_1^{(n-2)}(0) - p_1^{(n-2)}(0) + \int_0^x q_1(t)dt, \\
q_3(x) &= g_1^{(n-3)}(0) - p_1^{(n-3)}(0) + \int_0^x q_2(t)dt, \\
&\cdots \\
q_n(x) &= g_1(0) - p_1(0) + \int_0^x q_{n-1}(t)dt.
\end{aligned}$$

It follows from this construction that $\| q_2 - g_1^{(n-2)} + p_1^{(n-2)} \| < \varepsilon/n$, $\| q_3 - g_1^{(n-3)} + p_1^{(n-3)} \| < \varepsilon/n, \ldots, \| q_n - g_1 + p_1 \| < \varepsilon/n$, and hence

$$(**) \qquad \ll q_n - g_1 + p_1 \gg_{n-1} < \varepsilon.$$

Moreover, q_n is a nearly polynomial function because q_1 is a nearly constant function, so $q_n + p_1$ is a nearly polynomial function. Thus $q_n^{(n)} + p_1^{(n)} = p_1^{(n)}$ almost everywhere on I. Put $f_1 = q_n + p_1$. It follows from (*) and (**) that $| f_1^{(n)}(x) - g_0(x) | < \varepsilon$ almost everywhere on I and $\ll f_1 - g_1 \gg_{n-1} < \varepsilon$.

Proof (ii). Use the Weierstrass Approximation Theorem to find a polynomial g_1 such that $\| g_2 - g_1 \| < \frac{1}{2}\varepsilon$. Use part (i) with $\frac{1}{2}\varepsilon$ in place of ε to find a nearly polynomial function $f_2 \in C_{n-1}(I)$ so that $\ll f_2 - g_1 \gg_{n-1} < \frac{1}{2}\varepsilon$ and $| f_2^{(n)}(x) - g_0(x) | < \frac{1}{2}\varepsilon$ almost everywhere on I. It follows routinely that this function f_2 suffices for (ii). □

Let $p_1, p_2, p_3, \ldots$ be an enumeration of the polynomial functions in x on I with rational coefficients. These functions form a dense subset of $C(I)$ and of $C_n(I)$ for each integer n. In what follows m denotes Lebesgue measure.

Lemma 3. Fix an integer $n \geq 1$. Let k, i_1, i_2, i_3 be positive integers and let p_k be the polynomial in x in the enumeration mentioned before. Let $X(k, i_1, i_2, i_3)$ be the subset of $C(I)$ composed of functions f satisfying $m(E_t) \geq 1/i_1$ for all $t \in (0, 1/i_3)$ where

$$E_t = \{x \in I : | F_n(x,t)/t^n - p_k(x) | \geq 1/i_2\}.$$

Then

(i) $X(k, i_1, i_2, i_3) \cap C_{n-1}(I)$ is a closed nowhere dense subset of $C_{n-1}(I)$,

(ii) $X(k, i_1, i_2, i_3)$ is a closed nowhere dense subset of $C(I)$.

Proof. Let $g \in C_{n-1}(I)$ lie in the closure of $X(k, i_1, i_2, i_3)$ relative to $C_{n-1}(I)$. Fix $t_0 \in (0, 1/i_3)$. There is a sequence $(f_j) \subset X(k, i_1, i_2, i_3) \cap C_{n-1}(I)$ converging to g in $C_{n-1}(I)$ (and hence in $C(I)$) such that $m(E_{j,t_0}) \geq 1/i_1$ for each j where $E_{j,t_0} = \{x \in I : | F_{j,n}(x, t_0)/t_0^n - p_k(x) | \geq 1/i_2\}$ and where $F_{j,n}$ is formed from f_j the same way as F_n was formed from f before. Now $G_n(x, t_0)$ is a linear combination of a finite number of functions of the form $g(x + it_0)$ $(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$. But $f_j(x)$ converges uniformly to $g(x)$ and $F_{j,n}(x, t_0)$ converges uniformly to $G_n(x, t_0)$ in x; it follows that

$$\lim_{j \to \infty} F_{j,n}(x, t_0)/t_0^n = G_n(x, t_0)/t_0^n \quad \text{uniformly in } x.$$

It follows that $m(S_{t_0}) \geq \limsup_{j \to \infty} m(E_{j,t_0}) \geq 1/i_1$ where

$$S_{t_0} = \{x \in I : | G_n(x, t_0)/t_0^n - p_k(x) | \geq 1/i_2\}.$$

Consequently $g \in X(k, i_1, i_2, i_3)$ and $X(k, i_1, i_2, i_3) \cap C_{n-1}(I)$ is a closed set in $C_{n-1}(I)$.

Again, let $g \in X(k, i_1, i_2, i_3) \cap C_{n-1}(I)$ and let $\varepsilon > 0$. By Lemma 2(i) there is a nearly polynomial function $q \in C_{n-1}(I)$ such that $\ll g - q \gg_{n-1} < \varepsilon$, $| q^{(n)}(x) - c_n^{-1} p_k(x) | < c_n^{-1}/i_2$ and $| c_n q^{(n)}(x) - p_k(x) | < 1/i_2$ almost everywhere on I. Because q is a nearly polynomial function, it follows from Lemma 1 that $\lim_{h \to 0} Q_n(x, h)/h^n = c_n q^{(n)}(x)$ almost everywhere on I. Consequently, $\limsup_{h \to 0} | Q_n(x, h)/h^n - p_k(x) | < 1/1_2$ almost everywhere on I. Finally, $m(U_t) < 1/i_1$ for some $t \in (0, 1/i_3)$ where

$$U_t = \{x \in I : | Q_n(x, t)/t^n - p_k(x) | \geq 1/i_2\}.$$

Thus $q \notin X(k, i_1, i_2, i_3)$ and $\ll q - g \gg_{n-1} < \varepsilon$. So $X(k, i_1, i_2, i_3) \cap C_{n-1}(I)$ is a closed nowhere dense subset of $C_{n-1}(I)$, and (i) is proved.

The proof of (ii) is the same with $\| g - q \|$ in place of $\ll g - q \gg_{n-1}$, and convergence in $C(I)$ instead of $C_{n-1}(I)$. So we leave it. □

Our results will be stated in two parts – one for $C_{n-1}(I)$ and the other for $C(I)$.

Theorem 1. Fix an integer $n \geq 1$. Then there is a residual set of functions f in $C_{n-1}(I)$ having the property: for each measurable real valued function g on I, there is a sequence of positive numbers (h_j) converging to 0, and depending only on h and n, such that $\lim_{j \to \infty} F_n(x, h_j)/h_j^n = g(x)$ almost everywhere on I.

Proof. Let $X(k, i_1, i_2, i_3)$ and p_k be as in Lemma 3 and let $X = \bigcup_{k, i_1, i_2, i_3} X(k, i_1, i_2, i_3)$. Then X is a first category subset of $C_{n-1}(I)$. Let $f \in C_{n-1}(I) \backslash X$. It suffices to prove that f satisfies the desired property.

For each $k \geq 1$, $f \notin X(k, 2^k, 2^k, 2^k)$. So there is a point $t_k \in (0, 2^{-k})$ such that $m(S_k) < 2^{-k}$ where

$$S_k = \{x \in I : | F_n(x, t_k)/t_k^n - p_k(x) | \geq 2^{-k}\}.$$

Now let g be a measurable function on I. Let (p_{k_j}) be a subsequence of (p_k) converging to g almost everywhere on I. For each k,

$$| F_n(x, t_k)/t_k^n - p_k(x) | < 2^{-k} \quad \text{for } x \in I \backslash S_k.$$

But $m(S_k \cup S_{k+1} \cup S_{k+1} \cup \ldots) < 2^{1-k}$ and $m(\bigcap_{k=1}^{\infty} (S_k \cup S_{k+1} \cup S_{k+2} \cup \ldots)) = 0$. It follows that

(1)
$$\lim_{k\to\infty} [F_n(x, t_k)/t_k^n - p_k(x)] = 0$$

almost everywhere on I. Also,

(2)
$$\lim_{j\to\infty} [p_{k_j}(x) - g(x)] = 0$$

almost everywhere on I. From (1) and (2) we obtain

$$\lim_{j\to\infty} [F_n(x, t_{k_j})/t_{k_j}^n - g(x)] = 0$$

almost everywhere on I.

So $h_j = t_{k_j}$ suffices. □

Theorem 2. There is a residual set of functions f in $C(I)$ satisfying the property: for each measurable real valued function g on I and each integer $n \geq 1$, there is a sequence of positive numbers (h_j) converging to 0, and depending only on g and n, such that

$$\lim_{j\to\infty} F_n(x, h_j)/h_j^n = g(x) \quad \text{almost everywhere on } I.$$

Proof. The plan is to fix n and find an appropriate residual subset of $C(I)$ for n. But this argument is just like the proof of Theorem 1, so we leave it. □

In [1] Marcinkiewicz proved a little more than the case $n = 1$ in Theorem 2. The role of F_n in Theorem 2 can be played by certain other linear combinations of functions of the form $f(x + jh)$ $(j = 0, \pm 1, \pm 2, \pm 3, \ldots)$ over the integers.

Theorem 3. Fix an integer $n \geq 1$. Let c be a nonzero constant, and for any function f let $F(x,h)$ be a formal linear combination of functions of the form $f(x+jh)$ $(j = 0, \pm 1, \pm 2, \pm 3, \ldots)$ over the integers, such that for any polynomial function p, $\lim_{h \to 0} P(x,h)/h^n = cp^{(n)}(x)$. Then there is a residual set of functions f in $C(I)$ satisfying the property: for each measurable real valued function g on I, there is a sequence of positive numbers (h_j) converging to 0, and depending only on g, such that $\lim_{j \to \infty} F(x,h_j)/h_j^n = g(x)$ almost everywhere on I.

Proof. The proof of Theorem 3 is just like the development of Theorems 1 and 2. So we leave it. □

References

[1] J. Marcinkiewicz, Sur les nombres dérivés, Fundam. Math. 24 (1935) 305-308.

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