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ON THE EQUIVALENCE OF HENSTOCK-KURZWEIL AND RESTRICTED DENJOY INTEGRALS IN R^n

1. Introduction

Several descriptive definitions of the restricted Denjoy integral in R^n were given in the 1930's and 1940's [3, 8, 9], which are in terms of the generalized absolute continuity of the primitive function. However, for the next forty years, none of these has been proved to be equivalent to the Henstock-Kurzweil integral or Perron integral in R^n, except for the case when $n = 1$ [4, 6, 12]. For a recent attempt, see [10, p.83]. In this note, we shall settle the above problem, and, as a consequence, the problem posed by Pfeffer in [16, Problem 6.6]. Pfeffer's problem is: Give a Denjoy type descriptive definition of HF-integrals defined in section 4.

The Henstock-Kurzweil integral is of Riemann-type, which is defined by simply replacing the fixed norm δ in the Riemann integral by a positive function $\delta(x)$. This basic idea of replacing the constant δ of the classical definition by a positive function has been explored in many fields, for example, in integration theory [1, 6, 11, 15], in variation theory [5, 18] and in covering theory [2, 18]. The generalized absolute continuity of this type, denoted by ACG^*, is defined by Henstock in [6, p.58], which was drawn to the author's attention by P. Y. Lee. This Henstock version of ACG^* is equivalent to the classical ACG_* [17, Chapter VII] in the one-dimensional space R. With this Henstock version, the equivalence of the Henstock-Kurzweil and the restricted Denjoy integrals in R^n can be proved easily. Furthermore, many proofs in R can be shortened. The Henstock version is the genuine one whereas the classical version in R is the resultant of the Henstock version and the property of a closed set (i.e., the endpoints of a contiguous interval to a closed set are in the closed set itself). This is not the case for higher dimensional space. The restriction of the classical ACG_* was also pointed out by Henstock in [M.R.87b, #26010].

2. Generalized absolute continuity

Let R^n be the n-dimensional Euclidean space and E a nondegenerate closed interval $\prod_{i=1}^n [a_i, b_i]$ with a_i, $b_i \in R$ and $a_i < b_i$. Let Ψ be the class of all nondegenerate closed subintervals of E and let F be a real-valued interval function defined on Ψ. Given a closed interval I, let $r(I) = |I|/[d(I)]^n$ where d(I) and |I| denote the diameter and outer measure of I respectively. Given positive functions $\delta(x)$ and $0 < \rho(x) < 1$ defined on E, a partial partition $\{I_i\}$ of E with associate points $x_i \in I_i$ is said to be δ-fine if $d(I_i) \le \delta(x_i)$, and ρ-regular if $r(I_i) \ge \rho(x_i)$. Note that the regularity of I_i depends on x_i, i.e., the regularity of the partition is controlled by a positive function $\rho(x)$ instead of a constant. This idea is due to Pfeffer [15].

Now we consider the case when $n = 1$ and give the classical generalized absolute continuity in R [17, Chapter VII]. Let $X \subseteq [a,b]$. An interval function F is said to be $AC_*(X)$ if for every $\epsilon > 0$, there is $\eta > 0$ such that for any partial partition $\{I_i\}$ with $\Sigma|I_i| \le \eta$ and the endpoints of I_i belonging to X, we have

$$\Sigma\omega(F;I_i) \le \epsilon$$

where ω denotes the oscillation of F over I_i and $|I_i|$ the measure of I_i. Furthermore, F is said to be ACG_* on [a,b] if F is continuous and [a,b] is the union of a sequence of X_i such that on each X_i the function F is $AC_*(X_i)$. Similarly we define $BV_*(X)$ and BVG_*.

Next we shall give a modified version of ACG_* for $n = 1$. An interval function F is said to be $AC_{**}(X)$ if the condition "the endpoints of I_i belonging to X" in $AC_*(X)$ is replaced by the condition : "at least one of the endpoints of I_i belonging to X". In this definition, the oscillation of F is redundant and may be replaced by the difference of F. Similarly, we define ACG_{**}.

In the one-dimensional space R, the continuity of an interval function F is defined in the usual way [6, p.32]. That is, F is continuous at x in R if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever I is a closed interval containing x with $|I| \leq \delta$, we have $|F(I)| \leq \epsilon$. An additive interval function F in [a,b] is a function F of subintervals of [a,b] such that $F(B) = \Sigma F(I)$, for each interval $B \subseteq [a,b]$ and each partition $\{I\}$ of B.

LEMMA 1. *Let* $X \subseteq [a,b] \subseteq R$. *Then* $AC_*(X)$ *and* $AC_{**}(X)$ *are equivalent provided that* F *is continuous and additive.*

PROOF. We shall only prove that $AC_*(X)$ implies $AC_{**}(X)$. The converse is obvious. Let F be $AC_*(X)$. We may assume X is closed, in view of the continuity of F. Let $\{[x_i,y_i]\}$ be the contiguous intervals to X. Note that x_i, $y_i \in X$ except perhaps the case when $x_i = a$ or $y_i = b$. Thus for every $\epsilon > 0$, there exists a natural number N such that

$$\sum_{i=N}^{\infty} \omega(F;\ [x_i,y_i]) \leq \epsilon.$$

In view of the continuity of F, there exists $\eta_1 > 0$ such that for any partial partition $\{I_i\}$ with $\Sigma|I_i| \leq \eta_1$ and at least one of the endpoints of I_i being in the set $\{x_i,\ y_i;\ i = 1, 2, \ldots, N-1\}$, we have

$$\Sigma|F(I_i)| \leq \epsilon.$$

Let $\eta > 0$ be the number as obtained in the definition of $AC_*(X)$, when ϵ is given. We may assume $\eta < \eta_1$. Now take a partial partition $\{[a_i,b_i]\}$ with the total length less than η with a_i or b_i belonging to X for every i. For example, if $a_i \in X$, then b_i either belongs to X or lies in (x_k,y_k) for some k. Considering various cases and using the above two inequalities and the definition of $AC_*(X)$, we obtain

$$\sum_i |F([a_i,b_i])| \leq 4\epsilon.$$

That is, F is $AC_{**}(X)$. Note that in the above inequality, 2ϵ comes from the first inequality, one ϵ from the second inequality and the last ϵ from $AC_*(X)$. Also note that the additivity of F is used in the above inequality.

Finally, we shall give the Henstock version of the generalized absolute continuity in R^n [6, p.58]. Let $X \subseteq E \subseteq R^n$. An interval function F is said to be $AC^*(X)$ if for every $\epsilon > 0$, there exist positive functions $\delta(x)$ and $0 < \rho(x) < 1$ defined on E and $\eta > 0$ such that for any ρ-regular, δ-fine partial partition $\{I_i\}$ of E with associate points x_i in $I_i \cap X$ and $\Sigma|I_i| \le \eta$ we have $\Sigma|F(I_i)| \le \epsilon$. Similarly we define $BV^*(X)$, ACG^* and BVG^* [6, p.46; 18, p.38]. We remark that the above form (due to P. Y. Lee) is different from Henstock's, however they are equivalent [6, p.58, line 15, p.47, lines 17–21]. Note that we impose the regularity on I_i here whereas Henstock does not. For one-dimensional case, the regularity is superfluous and F is continuous if F is ACG^*.

LEMMA 2. *In R, the definitions of ACG_*, ACG_{**} and ACG^* are equivalent provided that F is continuous and additive.*

PROOF. It is obvious that if F is ACG_{**} on [a,b], then F is ACG^* on [a,b]. In view of Lemma 1, it remains to prove that if F is ACG^* on [a,b], then F is ACG_* on [a,b]. It is known that if F is ACG^*, then F is BVG^* [6, p.58] and, BVG^* and BVG_* are equivalent if F is continuous [18, p.94 Theorem (40.1); 5]. Thus F is BVG_* if F is ACG^*. Now we shall prove that F fulfills Lusin's condition (N), i.e., $|F(Q)| = 0$ if $|Q| = 0$ by using the ideas in [18, p.101]. Let $B_i = Q \cap X_i$, where F is $AC^*(X_i)$ and $\cup_i X_i = [a,b]$. Since F is $AC^*(X_i)$, therefore F is $AC^*(Q \cap X_i)$. Given $\epsilon > 0$, there exist $\delta(x) > 0$ and $\eta > 0$ such that $\Sigma|F(I)| \le \epsilon$ whenever $\{I\}$ is a finite sequence of nonoverlapping δ–fine intervals with associate points in $Q \cap X_i$ and $\Sigma|I| \le \eta$. On the other hand, $|Q \cap X_i| = 0$, thus there exists an open set G such that $Q \cap X_i \subseteq G$ and $|G| \le \eta$. Note that we may choose $\delta(x) > 0$ such that δ–fine intervals are always subsets of G. Therefore $\Sigma|F(I)| \le \epsilon$ for any finite sequence of nonoverlapping δ–fine intervals I with associate points in $Q \cap X_i$. Let $B_{in} = \{x \in B_i;\ \delta(x) > \frac{1}{n}\}$, $n = 1,2,\ldots$ and $I_{nm} = [\frac{m}{n}, \frac{m+1}{n})$, $m = 0, \pm1, \pm2, \ldots$. Obviously $B_i = \cup_n B_{in}$, $Q = \cup_i B_i$. Let $x,y \in B_{in} \cap I_{nm}$. Then [x,y] is a δ–fine interval with an associate point $x \in B_{in} \subseteq Q \cap X_i$. Hence $|F(B_{in})| \le \Sigma_{m=-\infty}^{\infty} |F(B_{in} \cap I_{nm})| \le \sup \Sigma|F(I)|$ where the supremum is taken over all finite sequences of nonoverlapping δ–fine intervals I with associate points in $Q \cap X_i$ [18, p.101, line 25]. Therefore $|F(B_{in})| \le \sup \Sigma|F(I)| \le \epsilon$ for every n. Then $|F(B_i)| \le \epsilon$, by taking the limit for the expanding sequence $\{F(B_{in})\}_n$ [18, p.101, lines 26–29].

Consequently $|F(Q)| = 0$. Therefore F fulfills Lusin's (N) condition. Hence F is ACG_* [18, p.106; 17, Theorem 6.8].

3. The restricted Denjoy integral

An interval function F defined on subintervals of E is said to be differentiable at a point x in E with derivative denoted by $F'(x)$ if for every $\epsilon > 0$, there exist $\delta(x) > 0$ and $0 < \rho(x) < 1$ such that whenever I is ρ-regular, δ-fine interval with an associate point $x \in I$, we have

$$|F(I) - F'(x)\ |I|| \le \epsilon |I|.$$

Now we shall give a descriptive definition of Denjoy's special integral for R^n. A function f defined on E is said to be restricted Denjoy integrable on E if there is an additive interval function F which is ACG^* on E and $F'(x) = f(x)$ for almost all x in E. An additive interval function F in E is a function F of subintervals of E such that $F(B) = \Sigma F(I)$, for each interval $B \subseteq E$ and each partition $\{I\}$ of B.

A function f is said to be Henstock-Kurzweil integrable on $E \subseteq R^n$ if there exists a number A such that for every $\epsilon > 0$, there exist $\delta(x) > 0$ and $0 < \rho(x) < 1$ such that

$$|\Sigma f(x)|I| - A| \le \epsilon$$

whenever $\{I\}$ is a ρ-regular, δ-fine partition of E with associate points $x \in I$ and Σ sums over $\{I\}$. Denote $A = \int_E f$.

It is easy to check that f is Henstock-Kurzweil integrable on each subinterval of E if f is Henstock-Kurzweil integrable on E [15, 3.4]. Denote $F(B) = \int_B f$ for each interval $B \subseteq E$. F is called the primitive of f. We shall discuss the additivity property of F with respect to the domain of integration in Section 4.

THEOREM 1. *A function f is Henstock-Kurzweil integrable on E iff f is restricted Denjoy integrable there.*

PROOF. (Necessity) Let f be Henstock-Kurzweil integrable on E and F the primitive of f. Then $F'(x) = f(x)$ almost everywhere [6, p.78; 12, p.46; the proof in [12] is valid in R^n, see [15], Proposition 4.4]. It remains to show that F is ACG^*. In fact, F is $AC^*(X_n)$ for every n, where $X_n = \{x; |f(x)| \le n\}$. This follows immediately from the following inequality

$$\Sigma|F(I)| \le \Sigma|F(I) - f(x)|I|| + \Sigma|f(x)||I|$$

The first term on the right-hand side of the inequality is less than ϵ, in view of Saks-Henstock Lemma [13, Lemma 1; 15, 4.3]. The second term is less than ϵ if $\Sigma|I| < \epsilon/n$. See also [6, p.59].

(Sufficiency) It is analogous to the one-dimensional case [4, 12]. Let f be restricted Denjoy integrable on E with primitive F. Note that F is ACG^* and $F'(x) = f(x)$ everywhere except in a set S of measure zero. For $x \in E - S$, given $\epsilon > 0$, there exist $\delta(x) > 0$ and $0 < \rho(x) < 1$ such that whenever I is ρ-regular, δ-fine interval with an associate point $x \in I$, we have

$$|F(I) - f(x)|I|| \le \epsilon|I|.$$

Since F is ACG^*, there exists a sequence of sets X_i such that $\cup_i X_i = E$ and F is $AC^*(X_i)$ for each i. Let $Y_1 = X_1$, $Y_i = X_i \setminus (X_1 \cup X_2 \cup \ldots \cup X_{i-1})$ for $i \ge 2$ and S_{ij} denote the set of points $x \in S \cap Y_i$ such that $j-1 \le |f(x)| < j$. Obviously, S_{ij}, $i,j = 1,2,\ldots$ are pairwise disjoint and their union is the set S. Since F is also $AC^*(S_{ij})$, there exist $\delta(x) > 0$ and $0 < \rho(x) < 1$ defined on S_{ij} and $\eta_{ij} < \epsilon\, 2^{-i-j}\, j^{-1}$ such that for any ρ-regular, δ-fine partial partition $\{I_k\}$ of E with associate points x_k in $S_{ij} \cap I_k$ and satisfying

$$\sum_i |I_k| < \eta_{ij} \quad \text{we have} \quad \sum_k |F(I_k)| < \epsilon\, 2^{-i-j}.$$

Choose G_{ij} to be the union of a sequence of open intervals such that

$$|G_{ij}| < \eta_{ij} \qquad \text{and} \qquad G_{ij} \supset S_{ij}$$

where $|G_{ij}|$ denotes the total length of G_{ij}. Now for $x \in S_{ij}$, $i,j = 1,2,\ldots$, we may redefine $\delta(x)$ such that whenever I is δ-fine interval with an associate point x, we have $I \subseteq G_{ij}$.

Take any ρ-regular, δ-fine partition $\{I\}$ with associate points $\{x\}$. Split the sum Σ over $\{I\}$ into two partial sums Σ^1 and Σ^2 in which $x \notin S$ and $x \in S$ respectively and we obtain

$$|F(E) - \Sigma f(x)|I|| \le \Sigma^1|F(I) - f(x)|I|| + \Sigma^2|F(I)| + \Sigma^2|f(x)|I||$$

$$< \epsilon\ |E| + \sum_{i,j} \epsilon\ 2^{-i-j} + \sum_{i,j} j\ \eta_{ij}$$

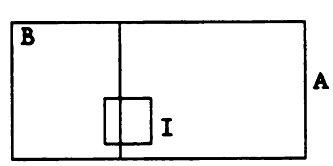
$$< \epsilon\ |E| + 2\epsilon.$$

Thus f is Henstock-Kurzweil integrable to F(E) on E.

The above proof is very much simpler in view of the one-point instead of two-points definition of ACG^*. This remarkable simplification can also be done in many proofs, for example, the proof of the controlled convergence theorem [13].

4. The additivity property with respect to the domain of integration

The integrals defined in Section 3 do not have the additivity property with respect to the domain of integration, i.e., the integrability over each member of a finite division of an interval does not imply the integrability over the whole interval. For a counterexample, see [7, Example 1; 15, Example 7.2]. This is due to the fact that if I is a ρ-regular subinterval of an interval A, then, in general, $I \cap B$ is not necessarily a ρ-regular subinterval of B. See the following diagram.



We may overcome this undesirable property by using integrals defined by Pfeffer [15] or Jarnik, Kurzweil and Schwabik [7]. Here we shall only give an integral defined by Pfeffer [15, 7.7] with a slight modification to illustrate the idea.

A function f is said to be HF-integrable on E if there exists a number A such that for every $\epsilon > 0$ and every finite family $\mathcal{H}$ of planes, there exist $\delta(x) > 0$ and $0 < \rho(x) < 1$ such that

$$\left|\Sigma f(x)|I| - A\right| \le \epsilon$$

whenever $\{I\}$ is a $(\rho,\mathcal{H})$-regular, δ-fine partition of E with associate points $x \in I$ and Σ sums over $\{I\}$. An interval I is said to be $(\rho,\mathcal{H})$-regular if $r(I,\mathcal{H}) > \rho(x)$. The regularity $r(I,\mathcal{H})$ of I relative to $\mathcal{H}$ is defined by Pfeffer [15, p.667] as follows : If $\mathcal{H} = \phi$, then $r(I,\mathcal{H}) = r(I)$; if $\mathcal{H}$ consists of a single k-plane H (k-dimensional linear submanifold of R^n), then $r(I,\mathcal{H}) = |I \cap H|_k/[d(I)]^k$ whenever $I \cap H \neq \emptyset$, and $r(I,\mathcal{H}) = r(I)$, otherwise, here $|I \cap H|_k$ denotes the k-dimensional outer measure of $I \cap H$. Finally, if $\mathcal{H} \neq \emptyset$ is arbitrary, then $r(I,\mathcal{H}) = \sup\{r(I,\{H\}); H \in \mathcal{H}\}$. For more detail, see [15].

HF-integrals have the additivity property as shown in the following theorem.

THEOREM 2 [15, 3.6]. Let f be a function on an interval E. Let $\mathcal{D}$ be a partition of E. If f is HF-integrable on D for each $D \in \mathcal{D}$, then f is HF-integrable on E and

$$\int_E f = \Sigma_{D\in\mathcal{D}} \int_D f.$$

Now we shall give a Denjoy type descriptive definition of HF-integrals, and hence settle the problem raised by Pfeffer in [16, Problem 6.6].

F is said to be $AC_{(p)}(X)$ if for every $\epsilon > 0$ and every finite family $\mathcal{H}$ of planes, there exist $\delta(x) > 0$, $0 < \rho(x) < 1$ and $\eta > 0$ such that for any $(\rho,\mathcal{H})$-regular, δ-fine partial partition $\{I\}$ with associate points $x \in I$ and $\Sigma|I| \le \eta$, we have $\Sigma|F(I)| \le \epsilon$. Similarly we define $ACG_{(p)}$.

A function f defined on E is said to be DF-integrable on E if there is an additive interval function F which is $ACG_{(p)}$ on E and $F'(x) = f(x)$ for almost all x in E.

Following the same ideas of the proof of the equivalence theorem in Section 3, we have

THEOREM 3. A function f is HF-integrable on E iff f is DF-integrable there on E.

We remark that we do not modify the definition of the derivative of F in the above integral since $|I \cap H| = 0$ for any plane H and if $x \notin H$ for each $H \in \mathcal{H}$, then we may define a ρ-regular, δ-fine interval I with an associate point $x \in I$ such that $I \cap H = \emptyset$ for each $H \in \mathcal{H}$.

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