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Separate and Joint Continuity II.

This is a continuation of my article [Pt]. Here, we pose some important open problems pertaining to separate versus joint continuity of functions defined on products of certain "nice" topological spaces.

In what follows let X, Y and Z be spaces and let a function $f:X\times Y\to Z$ be given. For every fixed $x \in X$, the function $f_x:Y\to Z$ defined by $f_x(y) = f(x,y)$, where $y \in Y$, is called an *x-section of f*. An y-section of f is defined similarly. We say that a function $f:X\times Y\to Z$ is *separately continuous* if f is continuous with respect to each variable while the other variable is fixed, i.e. if all of its x-sections f_x and y-sections f_y are continuous. Given a function $f:\prod_{i=1}^{n} X_i \to Z$; we shall denote that f is separately continuous by $f:\prod_{i=1}^{n} X_i \in\!\!\to Z$. Throughout this paper all the considered spaces are assumed to be Hausdorff.

§1. W. Sierpiński [Si] proved that if $X = Y = \mathbb{R}$ then every sepaately continuous function $f:X\times Y \in\!\!\to \mathbb{R}$ is uniquely determined by its values at the points of a dense subset D of the domain.

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This result is valid if the domain space is, roughly speaking, either:

a) both X and Y are metric and either X or Y is Baire [see Mc], or

b) if X is Baire and Y is second countable (see [GN] and [Co]).

Remark 1. It will be interesting to know "the size" (in various senses) and Borel class of the set D, in general case.

Remark 2. The "almost-continuity" condition for a function to be "uniquely determined by its values at the point of a dense subset D of the domain" also seems to be worthwhile of some deeper analysis, see for example [Ne].

Problem 1. *Characterize $\mathcal{X}$'s such that Sierpiński theorem holds, $\mathcal{Y}$ being compact.*

§2. R. Kershner [Ke] showed that the set D(f) of discontinuity points of any separately continuous function $f: \mathbb{R}^n \hookrightarrow \mathbb{R}$ has the dimension at most n-2. As we know, if X is separable metric, then ind X = Ind X = dim X, where ind, Ind and dim stand for the small inductive dimension, the large inductive dimension and the covering dimension, respectively.

Problem 2. *Let us assume that $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ are "nice" normal spaces and let $f: \prod_{i=1}^{n} \mathcal{X}_i \hookrightarrow \mathbb{R}$. Must Ind $\mathcal{D}(f) \le n-2$ (or dim $\mathcal{D}(f) \le n-2$)? In particular, is this true if $\mathcal{X}_i$'s are compact?*

Remark 3. It is worthwile to know that there have been studies of "the size" (in various senses) of D(f) for a separately continuous function f: $\mathbb{R}^2 \to \mathbb{R}$. In fact, G. C. Young, W. H. Young [YY] (see also [Pt] p. 296) showed that D(f) may be large in sense of cardinality - may be *uncountable* in *every* rectangle contained in the unit square. T. Tolstoff [To] constructed a function f: $\mathbb{R}^2 \hookrightarrow \mathbb{R}$ whose D(f) has a *positive* Lebesgue measure (!) being large in measure-theoretical sense.

§3. Following [SR] a space X is called *Namioka* if for any compact space Y and any metric Z:

(*) every separately continuous function $f: X \times Y \hookrightarrow Z$ there is a dense G_δ set $A \subset X$ s.t. $A \times Y \subset C(f)$, where C(f) stands for the set of points of (joint) continuity of f.

Remark 4. It has been shown [Ch] that a metric space Z in this definition can be replaced by the unit interval. However, an interesting question is how far can we go in relaxing the condition upon the range space Z (see an analogical problem for Blumberg spaces (compare §7), ([PS] and [BP])).

Remark 5. One cannot expect Z to be "too large" for if [Ch] p. 459 shown that even in the case when X = Y = [-1,1] (closed interval with Euclidean topology), there is a *compact* space Z namely Z = $C([-1,1]^2, [-1,1])$ equipped with the pointwise convergence topology, so that (*) fails.

The following problem constitutes essentially Problem 944 I recorded in the New Scottish Book (Wrocław, Poland) in 1978.

Problem 3. *Let $\mathfrak{X}$ be Namioka, $\mathfrak{Y}$ be compact and let $\mathfrak{Z}$ be any second countable, or more generally, a space having σ-disjoint base. Does (*) hold?*

Since it has been shown ([SR]) that all completely regular Namioka spaces are Baire and, obviously, in Baire spaces residual sets coincide with sets containing dense G_δ subsets, we can replace the condition "dense G_δ set A" in (*) by "residual set A" (for completely regular X's).

§4. R. Kershner [Ke] characterized the set D(f) of discontinuity points of f: $\mathbb{R}^2 \hookrightarrow \mathbb{R}$, namely

Let $S \subset \mathbb{R}^2$. Then S is D(f) of a certain function f: $\mathbb{R}^2 \hookrightarrow \mathbb{R}$ iff S is an F_σ contained in the product of two sets of first category.

This result has been generalized to compact metric spaces, see [BN].

Problem 4. *Characterize $\mathfrak{D}(f)$ for functions $f\colon \mathfrak{X} \times \mathfrak{Y} \hookrightarrow \mathfrak{R}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are compact spaces.*

§5. It has been shown [SR] that all metric Baire spaces or separable Baire spaces are Namioka.

Problem 5. *What is a natural class of spaces containing all metric and all separable spaces such that Namioka and Baire spaces coincide?*

§6. In his remarkable paper [Na] I. Namioka asked (Remarks 1.3(b) p. 520) whether every - what we call now - Namioka space is Baire. The negative answer was provided by M. Talagrand [T2] see §7. In the same article the following spectacular problem was posed:

Problem 6. (M. Talagrand) *Let* $\mathcal{X}$ *be Baire,* $\mathcal{Y}$ *be compact and let* $f:\mathcal{X}\times\mathcal{Y} \hookrightarrow \mathbb{R}$. *Is* $\mathcal{C}(f) \neq \emptyset$?

Remark 6. If one assumes additionally in Problem 6 that Y is *first countable*, then the positive answer has been shown in [LP2] even for a larger class of functions $f:X \times Y \to \mathbb{R}$ namely, it is enough that all x-sections f_x are continuous (with the exception, possibly, of a first category set), and all its y-sections f_y are quasi-continuous (≡ inverse image of every open set in the range is contained between an open set and its closure in the domain space; such functions, as shown by S. Marcus, do not have to be Lebesgue measurable!) compare also [PW].

§7. Let us recall that a topological space X is called *Blumberg*[1)] if for every function $f: X \to \mathbb{R}$ there is a dense subset D of X such that f restricted to D is continuous (on D). It is known [BG] that for metric spaces:

1) In 1922 H. Blumberg showed that $\mathbb{R}$ has the mentioned property.

X is Blumberg iff X is Baire (iff X is Namioka, see [SR]).

H. E. White, Jr. [Wh] proved that there is a Baire space that is not Blumberg. M. Talagrand [T2] has showed that there is an α-favorable space (hence Baire) which is not Namioka.

If X or Y is a metrizable space then every $f: X \times Y \hookrightarrow \mathbb{R}$ is the pointwise limit of a sequence of continuous functions[1], we shall write then $f \in B_1(X \times Y)$. Consequently, if the pointwise compact subsets of C(X) are metrizable, then every $f: X \times Y \hookrightarrow \mathbb{R}$ belong to $B_1(X \times Y)$, Y being compact[2]. Very recently G. Vera [Ve] extended these results. Following him we will say that a topological space X is *Moran space* (see [Mo]) if every $f: X \times Y \hookrightarrow \mathbb{R}$ is in $B_1(X \times Y)$, Y being any compact space.

In view of §6, and the just presented material we have:

<u>Problem 7</u>. What are the relationships in the class of Baire spaces between Namioka, Blumberg, Moran, Sierpiński spaces (defined in Problem 1) and spaces X for which Talagrand's problem has a positive solution.

<u>Remark 7</u>. The question whether every Baire Moran space is Namioka was posed in by G. Vera [Ve] and has been answered, in positive, by him in his recent article "Vector-valued first Baire class functions".

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1) See [Ru], compare [En] and further discussion in [Pt] p. 299.

2) It happens, for example, if X is the support of some Borel measure and has a dense σ-compact subset [Ru].

§8. It is known ([CT], [B2]) that if Y is second countable, and M is metric, then:

(**) for every separately continuous function

$f: X \times Y \to M$ there exists a residual set

$A \subset X$ such that $A \times Y \subset C(f)$.

(i) Be aware of the fact, that if Y is *first countable* (even metric complete) and $M = \mathbb{R}$, then (**) does not have to be true even in the case if X is the closed unit interval [0,1]! - [B1] see [Pt], Ex. 6.14 p. 313.

(ii) Also, if the space Y is assumed only to have a *countable network*[1)], which implies that Y is hereditarily Lindelöf and hereditarily separable, then again (**) does not have to hold, (see [T1], Remark (b), p. 241, see also [LP1], comments following Example 1, p. 288); see also [Pt], Ex. 6.13 p. 311.

Following [LP1] we say that a space Y is *co-Namioka*[2)] if for every Namioka space X condition (*) of §3 holds.

[1)] A family $\mathfrak{N} = \{N_s\}_{s \in S}$ of subsets of a space X is called a *network* if for every $x \in X$ and for every neighborhood U of x, there is $s_0 \in S$, such that $x \in N_{s_0} \subset U$.

[2)] This term has been used independently by G. Debs in a different sense, namely to denote these Y's, such that for any Baire space X (*) holds. The class of Debs' co-Namioka spaces, denoted usually by N*, contains all Corson-compact spaces. Recently, R. Deville [De] showed that N* contains all the compacts $[0,\Gamma)$ (Γ-an ordinal), and all scattered compact K's such that $K^{(\Omega)} = \emptyset$, where Ω is the first uncountable ordinal. He asked also whether N* contain all scattered compact spaces.

Well, by the definition, compact spaces are co-Namioka. We have shown [LP1] Theorem, p. 289, that k_ω-spaces are co-Namioka rel (LC), LC denotes the class of locally compact spaces, that is; if X is any locally compact space, Y is a k_ω-space, then (*) of §3 is true.

Further, every locally compact σ-compact space is co-Namioka. It easily follows from [CT] and [B2] that all second countable spaces are co-Namioka.

The space Y of (i) serves as an example of a complete metric, locally compact space which is *not* co-Namioka.

Likewise, Y of (ii) illustrates that not all hereditarily Lindelöf and hereditarily separable spaces must be co-Namioka.

Problem 8. *Characterize co-Namioka spaces.*

§9. Although as yet the class of Namioka spaces has not been characterized (internally), there is a need for the determination of permanence properties of Namioka spaces. Some invariants have already been discovered in [HJT], however the following problem is still open.

Problem 9. (R. Hansell [H1]) *If $\mathfrak{X}$ is closed-hereditarily Baire and Namioka, is every nonempty closed subspace of $\mathfrak{X}$ Namioka? Are dense $\mathcal{G}_\delta$ subspaces of Namioka spaces Namioka? What other permanence properties Namioka spaces have?*

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