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THE CHI FUNCTIONS IN GENERALIZED SUMMABILITY

In [2], Baric defined, for conservative matrices, a generalized summability analogue to the chi functional of scalar summability. This function, together with analogues to the functionals χ_n, may be defined for any conservative transformation T. These functions have multiplicative properties similar to those established for the chi functionals in [12, section 3]. We use these properties to give a necessary and sufficient condition for an invertible conservative matrix to have a matrix inverse. We also use the chi functions to make algebraic statements about certain algebras of conservative matrices and to show that some of them are Banach algebras. We close with remarks on certain algebras which contain conull matrices.

Our notation and terminology are standard. Let E be a Banach space. The spaces $m(E)$, $c(E)$, and $c_0(E)$ consist, respectively, of bounded sequences in E, convergent sequences in E, and null sequences in E. If $E = \mathbf{C}$ is the complex numbers, we write m, c, and c_0. Each of these spaces is a Banach space under the norm $||x|| = sup||x_k||$, $x = \{x_k\}$ a sequence in E. The coordinate functions C_n defined by $C_n(x) = x_n$ are continuous on these spaces. Baric [2] calls these spaces FK spaces since they are Frechet spaces with continuous coordinates. The space $\ell^1(E)$ consists of those sequences x in $c(E)$ for which $\sum ||x_k||$ is finite.

Let F be a second Banach space. A continuous linear transformation T from $c(E)$ to $c(F)$ is called *conservative* . If T can be represented by an infinite matrix $A = (A_{nk})$, n and k positive integers, where each A_{nk} is a continuous linear transformation from E to F, then T is called a conservative *matrix*. Conservative matrices are characterized in [1, Proposition 1.2](the characterization is due originally to Robinson [9] and Melvin-Melvin [7]). The set of conservative matrix transformations from $c(E)$ to $c(F)$ is denoted by $\Gamma(E,F)$, or by $\Gamma(E)$ if $E = F$, or by Γ if $E = F = \mathbf{C}$. In order that the function $\chi(T)$ be defined, it is necessary that F, the space underlying the range of T, be weakly sequentially complete, denoted *wsc*. We will always assume that the space underlying the range of a conservative transformation is *wsc*.

Let $s(E)$ denote the set of sequences in E, and let $A = (A_{nk})$ be a matrix of linear transformations each of which is continuous from E to F. If x is in $s(E)$, we say that A *sums* x if $\lim_n \sum_k A_{nk}(x_k)$ exists in the topology of F, each series being convergent in the same topology. Call A *null-conservative* if A sums each x in $c_0(E)$. Such matrices are characterized in [1, Proposition 1.1](see also [9] and [7]).

If X and Y are Banach spaces, $B(X,Y)$ denotes the set of continuous linear transformations from X to Y. We write $B(X)$ if $X = Y$. Composition of functions is denoted by juxtaposition; X^* is the (continuous) dual of X. All sums are indexed from 1 to infinity unless otherwise specified. The end of a proof is denoted $\square$.

1 The chi functions

We alter Baric's notation slightly and define insertion functions $e : E \to s(E)$ by $e(x) = \{x, x, \ldots\}$ and $e^k : E \to s(E)$, k a positive integer, where $e^k(x)$ is the sequence with k-th coordinate x and all other coordinates zero. These functions are clearly linear. The function e is continuous into $c(E)$ and the e^k are continuous into $c_0(E)$. Thus, the insertion functions are continuous into $c(E)$, $m(E)$, and $s(E)$ since larger FK spaces have weaker topologies ([2, p.168] and [10, p.203]). These functions also have properties reminiscent of a Schauder basis as shown in the following result, which is essentially in [2].

Lemma 1.1 *Let $x = \{x_k\}$ be in $c(E)$. Then*

1. $x = e(\lim x) + \sum e^k(x_k - \lim x)$, *so that*
2. $x = \sum e^k(x_k)$ *if and only if x is in $c_0(E)$.*

Proposition 1.2 *Let X be an FK subspace of $c_0(E)$, and T a continuous linear transformation from X to $m(F)$. Then T can be represented by a matrix.*

Proof. For $x = \{x_k\}$ in X, use (1.1.2) to compute Tx. Use the continuity of C_n to compute C_nTx. Define the matrix in the obvious way.□

Thus, if T is conservative from $c(E)$ to $c(F)$, then the restriction of T to $c_0(E)$ is a null-conservative matrix $B = (T_{nk}) = (C_nTe^k)$ called the *matrix part* of T, which will be denoted $B[T]$ if confusion could arise.

Proposition 1.3 *Let T be conservative from $c(E)$ to $c(F)$.*

1. *if $x = \{x_k\}$ is in $s(E)$, the series $\sum_k T_{nk}(x_k)$ and $\sum_k T_k(x_k)$ converge weakly in F, where $T_k(x_k) = \lim_n T_{nk}(x_k)$;*

2. *the linear function S_n from E to F defined by $S_n(x) = \sum_k T_{nk}(x)$ is continuous, as is the linear function S from E to F defined by $S(x) = \sum T_k(x)$.*

Proof. To see (1), choose f in F^* and consider the scalar matrix $B' = (b_{nk})$, where $b_{nk} = fT_{nk}(x)$. Then B' maps c_0 to c. To see this, let $z = \{z_k\}$ be in c_0 and $y_n = \sum_k b_{nk}z_k$. But $zx = \{z_kx_k\}$ is in $c_0(E)$ and so $T(zx)$ is in $c(F)$. Hence, $\lim_n fC_nT(zx)$ exists. But

$$fC_nT(zx) = f(\sum_k C_nTe^k(z_kx_k))$$

and the series converges in the norm topology of F. By the linearity and continuity of f

$$fC_nT(zx) = \sum_k fT_{nk}(x_k)z_k = y_n.$$

Thus, $y = \{y_n\}$ is in c and B' is null-conservative in the classical sense. Consequently, $\sum_k fT_{nk}(x_k)$ is (absolutely) convergent for each n. Since F is *wsc*, $\sum_k T_{nk}(x_k)$ converges weakly in F for each n.

Since B is null-conservative, $\lim_n T_{nk}(x_k) = T_k(x_k)$ exists in F for each x in E. Therefore, $fT_k(x_k) = \lim_n fT_{nk}(x_k)$ exists for each k and $\sum |fT_k(x_k)|$ is finite since B' maps m to m. Therefore, the scalar series $\sum fT_k(x_k)$ converges and, as above, $\sum_k T_k(x_k)$ converges weakly in F.

To see (2), use [2, Proposition 3.2] on the partial sums. □

Observe that if $x = \{x_k\}$ is in $c_0(E)$, then the series $\sum_k C_nTe^k(x_k)$ converges in the norm topology of F since B is null-conservative.

Definition 1.4 *Let T be conservative from $c(E)$ to $c(F)$. The function $\chi(T)$ from E to F is defined by*

$$\chi(T)(x) = \lim Te(x) - \sum \lim Te^k(x)$$

and the function $\chi_n(T)$ from E to F is defined by

$$\chi_n(T)(x) = C_nTe(x) - \sum_k C_nTe^k(x),$$

n a positive integer.

The series converge weakly by the previous proposition with the sequence x replaced by the constant sequence $e(x)$, x in E. The functions are clearly linear. The linear function $\lim : c(F) \to F$ is continuous [2, Prop. 2.3], so each chi function is the difference of continuous functions and hence continuous. Of course, $C_nTe^k(x) = T_{nk}(x)$ and $\lim Te^k(x) = \lim_n T_{nk}(x) = T_k(x)$, where $B = (T_{nk})$ is the matrix part of T.

Let v be the $B(E,F)$-valued sequence $\{\chi_1(T), \chi_2(T), \ldots\}$ and define the function $v \otimes \lim$ from $c(E)$ to $m(F)$ by

$$(v \otimes \lim)(x) = \{\chi_1(T)(\lim x), \chi_2(T)(\lim x), \ldots\}.$$

To see that $v \otimes \lim$ maps to $m(F)$, note that if $y_n = \sum_k C_nTe^kx = \sum_k T_{nk}(x)$, then $\{y_n\}$ is weakly bounded because the scalar matrix $B' = (fT_{nk}(x))$ is null-conservative for each x in E and f in F^*. Thus, $\{y_n\}$ is bounded and so $||\chi_n(T)(x)|| \le ||T|| + \sup_n ||y_n|| < \infty$ for each x in E. Now use Banach-Steinhaus to obtain a uniform bound for the $||\chi_n(T)||$.

Theorem 1.5 *Let T be conservative from $c(E)$ to $c(F)$. Then T may be written as $T = v \otimes \lim + B$, where B is the matrix part of T.*

Proof. Since both T and C are continuous and linear

$$C_nTx = C_nTe(\lim x) - \sum_k C_nTe^k(\lim x) + \sum_k C_nTe^k(x_k)$$

$$= C_n(v \otimes \lim + B)(x). \square$$

Remark. This theorem generalizes a result of Crawford [6, p.34]. Evaluating T, first at $e^k(x)$ and then at $e(x)$ for x in E, we see that the sequence v and the matrix B are uniquely determined by T. The result is in the literature in [3,4,5] Evidently, T is a matrix if and only if $\chi_n(T) = 0$ for all n (compare [11, p.357]).

Proposition 1.6 *If T is conservative from $c(E)$ to $c(F)$ and $x = \{x_k\}$ is in $c(E)$, then*

1. $\lim Tx = \chi(T)(\lim x) + \sum_k T_k(x_k)$,
2. $C_n Tx = \chi_n(T)(\lim x) + \sum_k T_{nk}(x_k)$

the series converging weakly.

Proof. To see (1), let $x = \{x_k\}$ be in $c(E)$ and $l = \lim_k x_k$. Then

$$\begin{aligned}\lim Tx &= \lim Te(l) + \sum_k \lim Te^k(x_k - l)\\ &= \lim Te(l) - \sum_k \lim Te^k(l) + \sum_k T_k(x_k)\\ &= \chi(T)(l) + \sum_k T_k(x_k),\end{aligned}$$

the norm or weak convergence of the series being justified by (1.1.1) and (1.3). A similar argument establishes (2).□

2 Properties of the chi functions; applications

The results of this section parallel those in [12, section 3]. For f in $c(E)^*$, define $J(f)$ in E^* by

$$J(f)(x) = fe(x) - \sum fe^k(x) \tag{1}$$

where x is in E. The series converges since the sequence $\{fe^k\}$, k from 1 to infinity, represents the restriction of f to $c_0(E)$ and hence is in $c_0(E)^*$, which is congruent to $\ell^1(E^*)$ in such a way that $\sum_k \|fe^k\| < \infty$. If $x = \{x_k\}$ is in $c(E)$, (1.1.1) allows us to see that

$$f(x) = J(f)(\lim x) + \sum fe^k(x_k). \tag{2}$$

In special instances, the notation $J(f)$ may be extended to certain f in $B(c(E), F)$, for example $\chi(T) = J(\lim T)$ and $\chi_n(T) = J(C_n T)$. Proposition (1.6) shows that this extension is consistent with (2).

Lemma 2.1 *Let T be conservative from $c(E)$ to $c(F)$, f in $c(F)^*$, and x in E. Then*

$$J(fT)(x) = J(f)\chi(T)(x) + \sum fe^k(\chi_k(T)(x)).$$

Proof. Using (2), we see that if $x = \{x_k\}$ is in $c(E)$, then

$$f(Tx) = J(f)(\lim Tx) + \sum fe^k(C_kTx). \tag{3}$$

Replace both $\lim Tx$ and C_kTx using (1.6). Let x be in E and compute $J(fT)(x)$ directly from (1). The desired equality holds up to an additive factor of

$$\sum_k \sum_r fe^kT_{kr}(x) - \sum_r \sum_k fe^kT_{kr}(x).$$

Since the left hand sum converges absolutely, the iterated sums are equal and the equality holds.□

Lemma 2.2 *If T is conservative from $c(E)$ to $c(F)$, and S is conservative from $c(F)$ to $c(G)$, then*

1. $\chi(ST) = \chi(S)\chi(T) + \sum S_k\chi_k(T)$
2. $\chi_n(ST) = \chi_n(S)\chi(T) + \sum_k S_{nk}\chi_k(T)$.

Proof. Equation (3) is valid for both $f = \lim S$ and $f = C_nS$. The equations so obtained can be used to compute $\chi(ST)$ and $\chi_n(ST)$. To prove the lemma, it suffices to show that

$$\sum_k \sum_j S_kT_{kj}(x) = \sum_j \sum_k S_kT_{kj}(x)$$

and that

$$\sum_k \sum_j S_{nk}T_{kj}(x) = \sum_j \sum_k S_{nk}T_{kj}(x)$$

for all x in E and for each positive integer n. Both equalities follow by the argument on [2, page 175].□

Remark 2.3 *From (2.2), it is easy to show that if T is a matrix, or if S sends $c_0(F)$ to $c_0(G)$, then $\chi(ST) = \chi(S)\chi(T)$. If we let $E = F = G$, we see that χ is multiplicative on $\Gamma(E)$. With $E = G$, it follows that if T in $\Gamma(E,F)$ is invertible with inverse S in $\Gamma(F,E)$, then $\chi(T)$ in $B(E,F)$ is invertible with inverse $\chi(S)$ in $B(F,E)$.*

As usual, call a matrix A in $\Gamma(E,F)$ *conull* if $\chi(A) = 0$ (in $B(E,F)$) and let $\Gamma_0(E,F)$ denote the set of conull matrices. Using (2.2), it is possible to show that no conull matrix is invertible, that $\Gamma_0(E)$ is an ideal in $\Gamma(E)$, and that $\Gamma_0(E)$ is a left ideal in $B(c(E))$.

Again following the usual terminology, we will say that a matrix A in $\Gamma(E, F)$ is *coregular* if $\chi(A) \neq 0$. If $\chi(A)$ is also invertible, we will say that A is *strongly coregular*.

Theorem 2.4 *Let T in $\Gamma(E, F)$ be invertible with inverse S in $B(c(F), c(E))$. Then S is in $\Gamma(F, E)$ if and only if T is strongly coregular. If T is strongly coregular, then any left inverse for T is a matrix.*

Proof. If S is a matrix, then (2.3) implies $\chi(T)$ is invertible. Conversely, if T is a matrix, (2.2.2) shows that $\chi_n(ST) = \chi_n(S)\chi(T)$. But ST is a matrix, so the composition must be zero. Since $\chi(T)$ is invertible, $\chi_n(S) = 0$ for all n. Thus, S is a matrix and any left inverse for T is a matrix.□

Example 2.5 *It is easy to show that if T is in $\Gamma(E, F)$ and invertible with inverse in $B(c(F), c(E))$, then $\chi(T)$ must be one-to-one. Thus, if $E = F = \mathbf{C}^n$,$\mathbf{C}^n$ being complex n-space, then the algebra $\Gamma(E)$ is closed under inverses. If $E = \mathbf{C}^2$ and $F = \mathbf{C}$, then no matrix is invertible, since $\chi(T)$ cannot be one-to-one. However, if $E = \mathbf{C}$ and $F = \mathbf{C}^2$, then $\chi(T)$ could be one-to-one but not onto. Thus, there is the possibility of an invertible matrix with a nonmatrix inverse.*

Note that a matrix T from c to $c(\mathbf{C}^2)$ must have entries mapping $\mathbf{C}$ to $\mathbf{C}^2$, i.e. the T_{nk} are ordered pairs (a_{nk}, b_{nk}) of complex numbers and $T_{nk}(z) = (a_{nk}z, b_{nk}z)$. Let T be the matrix defined by $T_{n1} = (-1, 0)$, $T_{n,2n} = (1, 0)$, and $T_{n,2n+1} = (0, 1)$ where n is a positive integer, all other entries being $(0, 0)$. It is not difficult to show that T is one-to-one and onto from c to $c(\mathbf{C}^2)$. If T has S as its inverse and S is a matrix, then the entries of S are also pairs of complex numbers, i.e. $S_{nk} = (c_{nk}, d_{nk})$, with S_{nk} mapping $\mathbf{C}^2$ to $\mathbf{C}$ by $S_{nk}(z_1, z_2) = c_{nk}z_1 + d_{nk}z_2$. Computing $S(Te^k)$ for k greater than or equal to 2, we see that $S_{1k} = (0, 0)$ for all k. But then $S(Te) = e$ must have first coordinate 0. This contradiction shows that S is not a matrix. Note that $e = e(1)$ and $e^k = e^k(1)$.

Remark. To this point, the results we have proved are valid for Frechet spaces as well. In this setting, the characterization of conservative matrices is given in [8, Theorem 1]. Null-conservative matrices can be characterized using parts (i) and (iii) of that theorem. Attention should be paid to Remark 1 on page 367. Useful information on $c_0(E)^*$ is contained in [2, Proposition 2.9]. The discussion immediately preceding that result is valuable, as is the subsequent description of $c(E)^*$.

Theorem 2.6 *The function $\chi : B(c(E), c(F)) \to B(E, F)$ is linear, continuous, and onto, as are the functions χ_n.*

Proof. Linearity is clear. To show onto, let L be in $B(E, F)$ and define a matrix A by $A_{nn} = L$ with other entries 0.

For continuity, let $T(n)$ be a Cauchy sequence in $B(c(E), c(F))$ converging to 0. But

$$||\chi(T(n))|| \leq ||T(n)|| + ||B(n)||$$

where $B(n)$ is the matrix part of $T(n)$. Also, $||B(n)|| \to 0$ since $B(n)$ is the restriction of $T(n)$ to $c_0(E)$. Since $||T(n)|| \to 0$ as well, χ is continuous. Similar arguments apply for χ_n.$\square$

Remark. See [1, Theorem 1.1.a] for the definition of $||A||$, A a null-conservative matrix. It is clear that $||B(n)|| \to 0$ on $c_0(E)$. The convergence is valid on $c(E)$ as well since the norm is defined via finite sequences.

Let $\Omega(E)$ denote the set of T in $B(c(E))$ for which $\lim_n \chi_n(T)(x)$ exists for each x in E.

Theorem 2.7 *Both $\Gamma(E)$ and $\Omega(E)$ are Banach algebras.*

Proof. From (2.2.2) it is clear that $\Gamma(E)$ is a subalgebra of $B(c(E))$. Also, $\Gamma(E)$ is the intersection of the null spaces of the continuous linear transformations χ_n and hence is closed. Since $\Gamma(E)$ contains the identity of $B(c(E))$, it is a Banach algebra under the inherited norm, i.e. the usual norm on $\Gamma(E)$.

If S is in $\Omega(E)$, then the matrix part of S, $B = (S_{nk})$, must be conservative. If T is also in $\Omega(E)$, then by (2.2.2)

$$\chi_n(ST) = \chi_n(S)\chi(T) + \sum_k S_{nk}\chi_k(T).$$

Now, $\lim_n \chi_n(S)\chi(T)(x)$ exists for each x in E. Also, if x is in E, the sequence

$$v = \{\chi_1(T)(x), \chi_2(T)(x), \ldots\}$$

converges. Hence, the series represents $C_n Bv$. But Bv converges. Therefore, for each x in E, $\lim_n \chi_n(ST)(x)$ exists and $\Omega(E)$ is an algebra. Furthermore, $\Omega(E)$ contains the identity of $B(c(E))$. To show that $\Omega(E)$ is closed, let $\{T_k\}$ be a Cauchy sequence in $\Omega(E)$ with limit T in $B(c(E))$. The sequence

$$v_k = \{\chi_1(T_k)(x), \chi_2(T_k)(x), \ldots\}$$

converges for each x in E and for each k. Let

$$v = \{\chi_1(T)(x), \chi_2(T)(x), \ldots\}.$$

Note that if S is in $\Omega(E)$ and S has matrix part B, then

$$||\chi_n(S)|| \leq ||S|| + ||B||$$

for all n. If we let $S = T - T_k$, we see that the sequence $\{v_k\}$ converges in norm to v. Hence, T is in $\Omega(E)$, and $\Omega(E)$ is a Banach algebra under the inherited norm.□

3 The rho function

Define a linear transformation $\rho : \Omega(E, F) \to B(E, F)$ where $\rho(T)$ is the function given by

$$\rho(T)(x) = \chi(T)(x) - \lim_k \chi_k(T)(x)$$

for x in E. Note that ρ is defined on no larger set and that $\rho(v \otimes \lim) = 0$ for each v in $c(B(E, F))$. The function $\chi_\infty(T)$ which is the pointwise limit of the $\chi_n(T)$ is continuous and

$$||\chi_\infty(T)|| \leq \sup ||\chi_n(T)|| \leq ||T|| + ||B||$$

where B is the matrix part of T. An argument similar to the proof of (2.6) shows that ρ is continuous. It is clear that ρ is onto since the restriction of ρ to $\Gamma(E, F)$ is χ (again (2.6)).

Denote by $\Omega_0(E, F)$ the set of T in $\Omega(E, F)$ for which $\lim \chi_n(T) = 0$ pointwise. Any T in $\Omega(E, F)$ may be written as $T = T_1 + T_0$ where T_0 is in $\Omega_0(E, F)$ and T_1 is of the form $e(L) \otimes \lim$ for some L in $B(E, F)$. To see this, recall that $T = v \otimes \lim + B$ where $v_n = \chi_n(T)$. Let L be the pointwise limit of the $\chi_n(T)$ and let u be the sequence $v - e(L)$. Then $T_1 = e(L) \otimes \lim$ and $T_0 = u \otimes \lim + B$.

Proposition 3.1 *Let T be in $\Omega(E, F)$ and S in $\Omega(F, G)$. Then $\rho(ST) = \rho(S)\rho(T)$.*

Proof. A direct computation using (2.2) shows that for x in E

$$\rho(ST)(x) = \chi(S)\chi(T)(x) + \sum_k S_k \chi_k(T)(x) -$$
$$\lim_n \chi_n(S)\chi(T)(x) - \lim_n \sum_k S_{nk} \chi_k(T)(x).$$

But we may write $S = S_1 + S_0$ and $T = T_1 + T_0$ so that

$$ST = S_1T_1 + S_0T_1 + S_1T_0 + S_0T_0.$$

Since ρ is linear, the result need only be checked on each type of summand. The checks are not difficult, but the following facts are useful:

1. if T is in $\Omega(X,Y)$, then the matrix part of T is conservative;

2. if T is in $\Omega_0(X,Y)$, $\rho(T) = \chi(T) = \chi(B[T])$, and $\{\chi_k(T)(x)\}$, k from 1 to infinity, is in $c_0(Y)$ for each x in X;

3. if $T = e(L) \otimes \lim$, then $\chi_k(T) = \chi(T)$ for each k;

4. [1, Theorems 1, 2].□

We should probably also point out that $\rho(v \otimes \lim) = 0$ even if the $B(E,F)$ valued sequence v converges only pointwise. In any event, it turns out that $\rho(S_iT_j) = 0$ except when $i = j = 0$, in which case $\rho(S_0T_0) = \chi(B[S])\chi(B[T])$ just as in the scalar case.

An immediate corollary of the proposition is that ρ is multiplicative on $\Omega(E)$.

4 Subalgebras of $B(c(E))$ containing conull matrices

A main result of [4] is that the only subalgebras of $B(c)$ which contain Γ_0, the kernel of χ, are Γ_0, Γ, Ω, and $B(c)$ itself, whereas the only subalgebras containing $K(\rho)$, the kernel of ρ, are $K(\rho)$, Ω, and $B(c)$. In general, there are infinitely many distinct subalgebras of $\Gamma(E)$(resp. $\Omega(E)$) which contain $\Gamma_0(E)$ (resp. $K(\rho)$). If E is infinite dimensional, there are even infinite chains of such.

The functions $\chi : \Gamma(E) \to B(E)$ and $\rho : \Omega(E) \to B(E)$ are algebra homomorphisms onto $B(E)$. Hence, there is a one-to-one correspondence between the subalgebras of $B(E)$ and the subalgebras of $\Gamma(E)$ (resp. $\Omega(E)$) which contain $\Gamma_0(E)$ (resp. $K(\rho)$). If E is infinite dimensional, choose $x_1 \neq 0$ in E and define the left algebra ideal

$$I_1 = \{T \in B(E) : Tx_1 = 0\}.$$

Choose x_2 not in the linear span of x_1 and define

$$I_2 = \{T \in B(E) : Tx_1 = Tx_2 = 0\}.$$

Choosing x_3 not in the linear span of x_1 and x_2 and proceeding inductively, we obtain a nonterminating descending chain of left algebra ideals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

with all containments proper. This chain lifts via χ (resp. ρ) to a chain

$$J_1 \supset J_2 \supset J_3 \cdots$$

of subalgebras of $\Gamma(E)$ (resp. $\Omega(E)$) each of which contains $\Gamma_0(E)$ (resp. $K(\rho)$). Again, all containments are proper.

If E has finite dimension at least 2, then E is isomorphic to $\mathbf{C}^n$, $n =$ dim E, and $B(E)$ is isomorphic to $M(n;\mathbf{C})$, the algebra of $n \times n$ complex matrices. Since any subalgebra of $M(n;\mathbf{C})$ is also a vector subspace of $M(n;\mathbf{C})$, it is obvious that any chain of subalgebras of $M(n;\mathbf{C})$ has length at most $n+1$. However, even in this case there are infinitely many distinct subalgebras of $\Gamma(E)$ (resp. $\Omega(E)$) containing $\Gamma_0(E)$ (resp. $K(\rho)$). I wish to thank L. Childs for suggesting the following example.

Example 4.1 *Let $n = 2$ and consider the matrix $A = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$, where a is nonzero. Any such matrix A generates a (commutative) two dimensional subalgebra $< A >$ of $M(2;\mathbf{C})$. This algebra is algebraically isomorphic to $\mathbf{C}[x]/(p_A)$, where $\mathbf{C}[x]$ is the algebra of polynomials in one variable with complex coefficients, and (p_A) is the ideal generated by p_A, the characteristic polynomial of A. If a is nonzero, p_A is also the minimal polynomial of A. Thus, $\{I, A\}$ is a basis for the algebra generated by A considered as a vector space.*

Let A and B be matrices of the given type and let p_A and p_B be their respective characteristic polynomials. Then it is not difficult to show that the following are equivalent:

1. *$< A >$ is isomorphic to $< B >$;*

2. *$< A > = < B >$;*

3. *$p_A = p_B$;*

4. $A = B$.

Hence, there are infinitely many distinct subalgebras of this type (as well as infinitely many distinct isomorphism classes of such) in $M(2;\mathbf{C})$.

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