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Change of variable in the semigroup valued refinement integral

Abstract. The standard change of variable formula, for T measurable from a measure space $(S, \mathcal{S}, \lambda)$ to a measurable space $(\overline{S}, \overline{\mathcal{S}})$ on which a measurable $\bar{t}$ is defined (see e.g. [H]), $\int_S \bar{t}T\, d\lambda = \int_{\overline{S}} \bar{t}\, d(\lambda T^{-1})$, is developed for a semigroup valued refinement integral. (The integral on the right can be rewritten in "Jacobian" form, $\int_{\overline{S}} \bar{t} \frac{d\lambda T^{-1}}{d\bar{\lambda}}\, d\bar{\lambda}$, when λT^{-1} is differentiable with respect to a measure $\bar{\lambda}$ on $\overline{\mathcal{S}}$; the attempt in [DS] Lemma III.10.8 to develop the result by starting from $\bar{\lambda}$ is incorrect). This yields a result for the order-convergent integral of real-valued integrands against positive finitely additive measures in an Archimedian ordered vector space as well as a (correct) result for the [DS] Banach space valued integral of a vector valued integrand against a real-valued finitely additive measure.

Motivation.

In the old-style classical texts (e.g. E.W. Hobson, Theory of Functions, Dover 1957, chap. VII) the Lebesgue integral is defined in a stepwise manner: First, the integral of a bounded measurable (real-valued) function t over a set of finite measure is defined by partitioning the range of t into finitely many pairwise disjoint subintervals, forming the sums of the lower/upper bounds of these subintervals multiplied by the measures of their inverse images under t, and taking the common limit of these sums (shown to exist), as the maximum length of the subintervals tends to zero, for the value of the integral. An unbounded measurable t is broken into its positive and negative parts each of which is then truncated to yield bounded measurable functions; their integrals are required to be bounded which permits defining the integral as the limit of that of the truncations as the bound on t is lifted to ∞. Finally, the integral of a measurable function over a set of infinite measure is defined by restricting t to subsets of finite measure and requiring that its indefinite integral over these exist and converge as the subset expands to fill out the whole domain.

Only inessential alterations need be made in this formulation to convert it to a semigroup valued refinement integral. Observe that the integral of a positive unbounded function over a set of finite measure could just as well be taken as the limit of its indefinite integral over the measurable subsets on which it is bounded as these expand to fill out the domain. Indeed, integrability entails finiteness almost everywhere, thus that the sets on which $|t| \leq N$ have complements whose measure decreases to 0 with increasing N: thus the integral over these sets passes arbitrarily close to the integral of every truncation of t as N increases, hence converges to the full integral of t. This permits combining the limiting processes used for unbounded functions and domains of infinite measure: Let $\mathcal{B}(t)$ denote the subsets of finite measure on which t is bounded — this collection is an ideal of measurable subsets, hence updirected: just define the integral as the limit of the indefinite integrals over the subsets in $\mathcal{B}(t)$ as the subset increases.

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As regards the integrals over the elements of $\mathcal{B}(t)$, they can be obtained as limits in the monoid of subsets of the reals: the totality of these subsets admits not only an associative commutative addition (the sum of two subsets being the set of pairwise sums of their elements) but also a multiplication by individual reals. Instead of obtaining the integral as the common limit of the upper and lower Darboux sums over the inverse images of a partition of the range of t into subintervals, one could equally well obtain it as the limit, in the monoid of subsets equipped with filter convergence, of approximating sums formed from their forward images by t, each multiplied as before by the measure of the subinterval's inverse image. These approximating sums are now indexed by finite disjoint families of measurable subsets, which could be more general than inverse images of subintervals; however, the limiting process which defines the integral (as well as the definition of measurability) still uses the subintervals of the range. To be free of this dependence, take the limit of the sums along the net of all finite decompositions of the domain into arbitrary measurable subsets ordered by refinement, rather than along that into inverse images of subintervals ordered by maximum length. That the latter convergence entails the former follows because refining a decomposition replaces each term Xy in the approximating sum by a subset of $\sum Xy_i$, $\sum y_i = y$, which is in the same real intervals. (And one could define measurability of t as the existence of this limit in every set of finite measure on which t is bounded).

The semigroup valued refinement integral.

The initial cast of characters comprises $\mathcal{Z}$, $\mathcal{S}$, $\mathcal{X}$.

$\mathcal{Z}$ is a commutative (additively written) monoid in which the approximating sums are formed, equipped with a notion of convergence which governs the way these sums are to approach the integral in $\mathcal{Z}$.

$\mathcal{S}$ is a (Boolean) ring; set theoretic notation will be used for its operations. The approximating sums will be indexed by the finite disjoint families in $\mathcal{S}$ ordered by refinement: here $\{S_i\}$ *refines* $\{S'_j\}$ when every S'_j is the join of the S_i it contains. Since this order is directed (indeed, a semilattice), the approximating sums become a net whose limit (if it exists) in $\mathcal{Z}$, will be the integral.

The integrands are functions t from an ideal $\mathcal{B}(t)$ in $\mathcal{S}$ to a set $\mathcal{X}$, integrated against a function λ from $\mathcal{S}$ to $\mathcal{Z}^{\mathcal{X}}$. The composite $t\lambda$ is the $\mathcal{Z}$-valued function which at each $S \in \mathcal{B}(t)$ takes as value the image of $t(S)$ under $\lambda(S)$; when extended to send the finite disjoint families which partition a fixed S in $\mathcal{B}(t)$ to the sum of the values at the terms, it becomes a net in $\mathcal{Z}$ whose limit under refinement is declared to be $\int_S t \cdot \lambda$; when this $\mathcal{Z}$-valued function has a limit as S increases in $\mathcal{B}(t)$, this limit is taken to be the full integral $\int_{\mathcal{S}} t \cdot \lambda$.

The change of variable formula.

A new character now enters: T, an isotone map sending only $\emptyset$ on $\emptyset$, from $\mathcal{S}$ to another ring $\overline{\mathcal{S}}$ ($\mathcal{X}$ and $\mathcal{Z}$ remain the same). For an integrand $\bar{t}$ on $\overline{\mathcal{S}}$ and the previous λ on $\mathcal{S}$, it is desired to compare the integrals $\int_{\mathcal{S}} \bar{t}T \cdot \lambda$ and $\int_{\overline{\mathcal{S}}} \bar{t} \cdot \lambda T^{-1}$. Here $T^{-1}\overline{S}$ is defined as the largest S for which $TS \subset \overline{S}$, assumed to exist in $\mathcal{S}$ for every $\overline{S} \in \overline{\mathcal{S}}$ and to satisfy: $\overline{S} \subset TS$ only if $\subset TT^{-1}\overline{S}$ (T is *measurable* as a transformation; the more usual definition will be in effect in the Measurable integrand section below.)

Since T^{-1} preserves $\cap$ and $\emptyset$, it sends finite disjoint families in $\overline{\mathcal{S}}$ to such in $\mathcal{S}$ and one has, for any $\overline{S}$, $\int_{\overline{S}} \bar{t}TT^{-1} \cdot \lambda T^{-1} = \int_{T^{-1}\overline{S}} \bar{t}T \cdot \lambda$ (both integrals being limits along refinement in $\overline{\mathcal{S}}$) the existence of either side implying that of the other and equality. Each of these forms will next be converted, in the presence of suitable hypotheses, into that desired.

The argument.

$TS \subset \overline{S}$ only if $S \subset T^{-1}\overline{S}$, thus $TS \subset TT^{-1}\overline{S} \subset \overline{S}$, hence $T^{-1}(\overline{S} \setminus TT^{-1}\overline{S}) = \emptyset$; if $\lambda\emptyset$ is the zero function then the nets defining $\int_{\overline{S}} \bar{t} \cdot \lambda T^{-1}$ and $\int_{\overline{S}} \bar{t}TT^{-1} \cdot \lambda T^{-1}$, $\overline{S} \in \mathcal{B}(\bar{t})$, are finally equal (since $T^{-1}TT^{-1} = T^{-1}$) so that if convergence in $\mathcal{Z}$ is final subnet invariant (i.e. the same convergence in effect for a net as for its final subnets) the existence of either limit will imply that of the other and equality.

In general T^{-1} does not preserve join[1] so it must be postulated that $T^{-1}\overline{\mathcal{S}}$ is closed under difference in $\mathcal{S}$ — this makes T^{-1} $\setminus$, as well as $\cap$, preserving. One could then succeed in equating $\int_{T^{-1}\overline{S}} \bar{t}T \cdot \lambda$ calculated in $\mathcal{S}$ and in $\overline{\mathcal{S}}$ if λ were finitely additive: i.e. sent finite disjoint joins in $\mathcal{S}$ to (pointwise) functional sums (this may be construed to include $\lambda\emptyset = 0$). However, the application in view calls for a more elaborate setting: $\mathcal{Z}$ to be a partially ordered monoid and λ to be finitely additive to a monoid Y whose elements act from $\mathcal{X}$ to $\mathcal{Z}$ "subadditively": i.e. $X \sum y_i \leq \sum X y_i$; moreover, convergence in $\mathcal{Z}$ should be "order refinement closed": with any convergent net also every net eventually dominated by each of its elements should converge to the same limit. Then the existence of $\int_{T^{-1}\overline{S}} \bar{t}T \cdot \lambda$ in $\mathcal{S}$ will entail its existence in $\overline{\mathcal{S}}$ and equality as a consequence of those finite partitions of $T^{-1}\overline{S}$, which refine the partition consisting of their $T^{-1}TS$'s, being cofinal (in $\mathcal{S}$). Indeed, every S^* of the latter is the $T^{-1}T$ of every S of the former it contains — hence $TS^* = TS$ — whence $\sum \bar{t}TS \cdot \lambda S$: $S \subset S^* \geq \bar{t}TS^* \cdot \sum \lambda S = \bar{t}TS^* \cdot \lambda S^*$.

To see that such families are cofinal in $\mathcal{S}$, start with any finite partition $\{S\}$ of $T^{-1}\overline{S}$ in $\mathcal{S}$, disjointify its TS's in $\overline{\mathcal{S}}$ and then refine $\{S\}$ with T^{-1} of this disjointification, $\{\overline{S}\}$. This refinement has the desired property since one of these $\overline{S}$ meets a TS only if it is contained in it; hence measurable T sends every non-void $T^{-1}\overline{S} \cap S = T^{-1}\overline{S}$ on $\overline{S}$.

The end result.

With T measurable and $T^{-1}\overline{\mathcal{S}}$ closed under difference, λ finitely additive (including $\lambda\emptyset = 0$) to the additive monoid Y acting subadditively from $\mathcal{X}$ to $\mathcal{Z}$, convergence in $\mathcal{Z}$ closed for order refinement, $T^{-1}\mathcal{B}(\bar{t}) \subset$ and cofinal (under inclusion) in $\mathcal{B}(\bar{t}T)$, the existence of $\int_S \bar{t}T \cdot \lambda$ entails

$$\int_{\overline{S}} \bar{t} \cdot \lambda T^{-1} = \int_S \bar{t}T \cdot \lambda$$

Measurable integrands.

The usual treatment can be more closely patterned when the integrands are restricted and the transformations extended. Call an $X \in \mathcal{X}$ *Y-convex* if $\sum X y_i \leq$

[1] Take T to be the join-endomorphism on the power set on three elements which retracts it on that of two elements by sending the third element to their join.

(hence = by subadditivity) $X \sum y_i$ and an integrand $\bar{t}$ *measurable* if every partition of an $\overline{S}^* \in \mathcal{B}(\bar{t})$ splits into $\overline{S}$'s each of which is sent by $\bar{t}$ into a Y-convex X, while the remaining $\overline{S}'$ of the partition are sent into Y-convex X' so that $\sum X \cdot \lambda\overline{S}$ converges (in $\mathcal{Z}$, under refinement) while $\sum X' \cdot \lambda\overline{S}'$ converges to 0. When Y acts isotonely (in addition to subadditively) this entails integrability in $\mathcal{B}(\bar{t})$, by order refinement of convergence. Now measurability of a transformation may be construed in a wider setting: as an isotone map sending only $\emptyset$ on $\emptyset$, from $\mathcal{S}$ to another ring $\overline{\mathcal{R}}$ (on which $\bar{t}$ is defined and isotone) containing $\overline{\mathcal{S}}$, a subring of "measurables", of which only $T^{-1}\overline{\mathcal{S}} \subset \mathcal{S}$ is required. Then $\bar{t}T$ is measurable on every $T^{-1}\overline{S}$ with the value of its integral equal to that of $\bar{t}$'s over $\overline{S}$; hence, choosing for $\mathcal{B}(\bar{t}T)$ the ideal generated by $T^{-1}\mathcal{B}(\bar{t})$ in $\mathcal{S}$, one has $\bar{t}T$ also measurable and its integral over S equal to $\bar{t}$'s over $\overline{S}$ when either exists.

The end result for measurable integrands.

With $\bar{t}$ a measurable integrand and T a measurable transformation (in the present broader sense), $\mathcal{B}(\bar{t}T)$ the ideal generated by $T^{-1}\mathcal{B}(\bar{t})$ in $\mathcal{S}$, λ finitely additive to Y acting isotonely and subadditively, and covergence in $\mathcal{Z}$ order refinement closed: the existence of either integral entails that of the other and equality.

Specialization to power sets.

In the usual setting $\mathcal{S}$ is a ring of subsets of a set $\underline{S}$, $\mathcal{X}$ and $\mathcal{Z}$ are the power sets of sets $\underline{X}$ and $\underline{Z}$ respectively, the latter is again a commutative (additive) monoid — thus $\mathcal{Z}$ inherits this structure — equipped with a notion of refinement closed (say filter) convergence — this makes convergence of nets in $\mathcal{Z}$ (to elements of $\underline{Z}$, in which the integrals will now be required to exist) closed under containment refining; commutative monoid Y acts additively from $\underline{X}$ to $\underline{Z}$ (equivalently, the elements of $\underline{X}$ induce monoid morphisms from Y to $\underline{Z}$) which makes Y subadditive and isotone from $\mathcal{X}$ to $\mathcal{Z}$; T is a point map from $\underline{S}$ to $\overline{\underline{S}}$ — hence $T^{-1}\overline{\mathcal{S}}$ exists (and is closed under difference).

The Banach space integral of [DS].

The integrand is vector-valued, the measure scalar-valued — thus Y-convexity is the usual convexity. Observe that every (norm) bounded [**DS**]-measurable function, as in measure limit of a sequence of simple functions, is refinement integrable over every subset of finite variation; every[**DS**]-integrable function (i.e. the in measure limit of a mean Cauchy sequence of simple functions) is integrable over every finite variation subset, the integral over a set on which t is bounded coinciding with the refinement integral; and the [**DS**]-integral of every t is the limit of its integrals over the finite variation subsets on which it is bounded (as these subsets increase). These subsets constitute the $\mathcal{B}(t)$; by enclosing the t-images of the finite variation S of a sufficiently fine partition in convex X of small diameter or in X' of bounded diameter, one sees that every [**DS**]-measurable t is measurable. Indeed, [**DS**]-measurablility comes to being able to enclose the t-images of some of the S of sufficiently fine partitions in convex X of small diameter, while λ of the union of the remaining ones is small. The argument in the Measurable integrand section can be repeated for this measurability and yields the corresponding result for the [**DS**]-integral.

Specialization to ordered setting.

$\underline{X}$ and $\underline{Z}$ are now partially ordered, Y acts isotonely (as well as additively) from $\underline{X}$ to $\underline{Z}$, addition is isotone in $\underline{Z}$ which will be equipped with order-convergence (an order refinement closed filter convergence). In this set-up the existence of $\int_S t\cdot\lambda$ over an S on which t is bounded follows when there exist lower/upper bounds v_i/u_i for the values of t over the measurable subsets S_i of S such that the sums $\sum v_i\cdot\lambda S_i/\sum u_i\cdot\lambda S_i$ have a common sup/inf under refinement (alternatively, one could describe these sums as the integrals of appropriate "simple functions"). When $\underline{X}$ is boundedly complete one can take for v_i/u_i the inf/sup of t over S_i; the lower/upper sums then increase/decrease under refinement and when these totalities are not separated in $\underline{Z}$, one obtains for $\int_S t\cdot\lambda$ the common value of their sup/inf — or, if the latter does not exist, the cut they define in the MacNeille completion. When t for which this holds takes values in the subset of $\underline{X}$ sent by Y into "positive" elements — i.e. addition by which is increasing in $\underline{Z}$ — the indefinite integral $\int_S t\cdot\lambda$ becomes isotone in the S on which t is bounded; hence the full integral is their sup. This is also the sup of all the lower $\sum v_i\cdot\lambda S_i$ and will exist in a boundedly complete $\underline{Z}$ when they are bounded. These conditions are realized when X is a vector lattice and t^+ and $t^-$are separately (i.e. t is "absolutely")integrable, since the action of Y and addition in $\underline{Z}$ are isotone. It follows that the order integral as defined in [**MW**] is the refinement integral — there $\underline{X}$ is the reals and Y the positive cone of an Archimedean ordered vector space $\underline{Z}$, integrability for positive t being defined as boundedness of the integrals of any increasing sequence of simple functions converging pointwise to t. The sup of these integrals is the same for any such sequence, hence coincides with the sup of the lower $\sum v_i\cdot\lambda S_i$. Since no subset of $\underline{X}$ is Y-convex, there are no integrands measurable in the above sense; however, real-valued measurable functions are integrable over sets of finite measure on which they are bounded; this consequence of measurability could serve as the definition here. With this definition, the argument in the Measurable integrand section can be repeated and one obtains

The end result in the ordered setting.

With $\bar{t}$ real-valued measurable, λ finitely additive to the positive cone of an Archimedean ordered vector space, T a point map with $T^{-1}\overline{S}\subset S$, and $\mathcal{B}(\bar{t}T)$ the ideal generated by $T^{-1}\mathcal{B}(\bar{t})$, the existence (in the MacNeille completion) of either of $\int_{\overline{S}}\bar{t}\cdot\lambda T^{-1}$, $\int_S\bar{t}T\cdot\lambda$ entails that of the other and equality.

[**MW**] assumes the measure countably additive; the result would then hold with $\mathcal{B}(\bar{t}T)$ only in the σ-ideal generated by $T^{-1}\mathcal{B}(\bar{t})$.

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