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# THE ω-LIMIT SETS FOR SELF MAPS OF AN INTERVAL

Let $f$ be a function mapping $I = [0,1]$ into itself. A set $K \subset I$ is called an $\omega$-limit set for $f$ if there exists $x \in I$ such that $K$ is the cluster set of the sequence $\{f^n(x)\}$. (Here, as usual, $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n = 1,2,3, --$.) We write $\omega_f(x) = K$ to indicate $K$ is the $\omega$-limit set of $x$ under $f$.

For sufficiently well-behaved functions one finds that either there is a single set $K$ that serves as the $\omega$-limit set for almost all $x \in I$, or there is some form of chaotic behavior.

For the typical continuous $f$, no single set can serve as an $\omega$-limit set for almost all $x \in I$. In fact [ABL], there exists a set $K$ of full measure such that for every $x \in K$, $\omega_f(x)$ is a Cantor set $K_x$ -- but the sets $K_x$ are distinct and pairwise disjoint: if $x \neq y$, then $K_x \cap K_y = \phi$.

An $\omega$-limit set must be closed, but beyond that, what restrictions must apply? On the one hand, one finds in the literature all sorts of closed sets that can serve as $\omega$-limit sets for a continuous function. Finite sets give rise to periodic behavior, Cantor sets arise in various ways, as do countable closed sets and even intervals. On the other hand, an $\omega$-limit set with interior must consist of a finite union of intervals.

In addition, Šarkovskii [S], indicates that if an infinite $\omega$-limit set contains an isolated point, it must contain infinitely many isolated points.

This assertion of Šarkovskii's led several of us [ABCP] to try to obtain a characterization of those sets that can serve as $\omega$-limit sets of continuous functions.

Since no proof of Šarkovskii's assertion appeared in [S], we tried first to prove this assertion. But we were unable to rule out a certain scenario that led to a revealing example.

Consider first a rudimentary example. Let

$$f(x)=\begin{cases} 3x & on\ \left[0,\tfrac{1}{3}\right] \\ 1 & on\ \left[\tfrac{1}{3},\tfrac{2}{3}\right] \\ 3(1-x) & on\ \left[\tfrac{2}{3},1\right] \end{cases}.$$

To analyze the iterative behavior, we represent points by their ternary expansions,

$$x = .x_1x_2x_3-- \quad , \quad x_i = 0,1,2 .$$

It is easy to verify that for $x \in \left[0,\tfrac{1}{3}\right]$, $f(x) = .x_2x_3---$ while for $x \in \left[\tfrac{1}{3},\tfrac{2}{3}\right]$, $f(x) = 1$ $\left(so\ f^2(x)=0\right)$ and for $x \in \left[\tfrac{2}{3},1\right]$, $f(x) = .x_2^*x_3^*--$ where $x_i^* = 2 - x_i$.

Thus, if a ternary expansion of $x$ contains a 1, then for some $n$, $f^n(x) = f^{n+1}(x) = -- = 0$. All but countably many points of the Cantor set $C$ have unique expansions containing only 0's and 2's. Thus the orbits of most points of the set $C$ miss the interval $\left[\tfrac{1}{3},\tfrac{2}{3}\right]$ and one finds various sorts of $\omega-$limit sets within $C$: e.g. periodic orbits of all order, countable $\omega-$limit sets, perfect $\omega-$limit sets that are nowhere dense in $C$ and the entire set $C$.

If the ternary expansions of $x$ begins with a long block of 0's and 2's, followed by a 1, its orbit will appear for a while to move around $C$ in one of many ways, only suddenly to meet the fate of most orbits under $f$ -- absorption by the fixed point 0! If we can modify $f$ on $\left[\tfrac{1}{3},\tfrac{2}{3}\right]$ to perturb the orbit just a little, perhaps we can avoid this fate obtaining an $\omega-$limit set consisting of a nowhere dense perfect set in $\left[0,\tfrac{2}{3}\right]$ together with the isolated point 1.

Let $K$ consist of all points whose ternary expansions take the form

$$x = .1\,0a_2\,a_3\ -- \qquad (a_i=0,2).$$

Thus $K$ is a copy of $C$ in $\left[\tfrac{1}{3},\tfrac{4}{9}\right]$ : $K = \tfrac{1}{9}\ C + \tfrac{1}{3}$.

Let $S$ consist of those $x \in I$ with expansions of the form

$$x = .10A_1 1010A_2 1010A_3 10--$$

where each $A_i$ is a block of 0's and 2's. Let $g(x) = 1 - 3d(x,K)$, where $d(x,K)$ is the distance between $x$ and $K$.

Thus $g$ modifies the function $f$ by exhibiting "spikes" on the intervals contiguous to $K$ on $[\frac{1}{3},\frac{2}{3}]$. The perturbing effect of these spikes makes it possible to exhibit a great variety of iterative behavior.

Let $x \in S$, $x = .10A_1 1010A_2 1010A_3 --$. By noting the nearest point to $x$ in $K$ is $y = .10A_1 10\bar{0}$ and applying the simple arithmetic computations previously described, we find $f^{n_1}(x) = .10A_2 1010A_3 --$, $n_1 = 4 +$ (length of block $A_1$). This is the first stage at which the orbit of $x$ revisits the interval $[\frac{1}{3},\frac{2}{3}]$. In effect, the first "block" $10A_1 10$ has been dropped. Similarly, there is a sequence $\{n_k\}$ of positive integers such that $g^{n_k}(x) = .10A_{k+1} 1010A_{k+2} 10--$ represents the *kth* revisit of the orbit of $x$ to $[\frac{1}{3},\frac{2}{3}]$. By choosing the sequence $\{A_k\}$ in various ways we can realize various $\omega$-limit sets. For example, if $M_o$ is any nonempty closed subset in $K$, choose $A_k$ so that $\{g^{n_k}(x)\} = \{.10A_k 1010A_{k+1} 10--\}_{k=1}^{\infty}$ clusters exactly on $M_o$. In this case, the resulting $\omega$-limit set is $\omega_g(x) = \bigcup_{k=o}^{\infty} M_k \cup \{0,1\}$ where $M_k = g^{-k}(M_o)$. The details are not difficult and are presented for a slightly more complicated function in [ABCP].

In a sense $g$ represents a universal quality: given any nowhere dense nonempty compact set $M_o$, there is a point $x \in I$ such that $W_g(x) \cap [\frac{1}{3},\frac{4}{9}]$ is homeomorphic to $M_o$.

The preceding example suggests that any nowhere dense, nonempty, compact set can be an $\omega$-limit set for a continuous function. In [ABCP] we showed this to be the case. But for certain complicated sets, our construction required all limit points to be fixed points of the function.

Nonetheless, it seemed likely that by further "perturbations" on the sets $M_k$ appearing in the example, one could realize any infinite nowhere dense compact set as a homoclinic $\omega$ - limit set -- one each of whose members lands on a specified fixed point or fixed periodic orbit. J. Smítal saw how to abstract the concrete example. This gave rise to [BS] in which we show that, indeed, each infinite nowhere dense compact set $M$ is a homoclinic $\omega$ - limit set for some continuous $f$.

The function $g$ in our example can be smoothened so as to be $C^\infty$. But not all infinite nowhere dense compact sets are $\omega$ - limit sets for even differentiable functions. To see this let $Z$ and $P$ be Cantor sets, $Z \cap P = \phi$, $Z$ of measure 0, and $P$ of positive measure in any open interval it intersects. If $Z \cup P$ is an $\omega$ - limit set for a continuous $f$, there must be a portion of $Z$ that maps onto a portion of $P$. (This follows readily from the fact that an $\omega$ - limit set for a continuous $f$ must map onto itself.) But then $f$ maps a zero measure set onto a set of positive measure and therefore can't be differentiable.

We summarize the main results so far.

Theorem. Let $M$ be any nonempty nowhere dense closed subset of $I = [0,1]$. Then $M$ is an $\omega$ - limit set for some continuous function $f$. If $M$ is infinite, one can choose $f$ so that $M$ is homoclinic with respect to $f$.

Since a homoclinic $\omega$ - limit set has at most one fixed point, this theorem does not provide much information about the nature of possible fixed points within an $\omega$ - limit set. Some information is provided in [ABCP]. If $M$ is nowhere dense, $M = P \cup C$ with $P$ perfect and $C$ countable, and $P \subset \bar{C}$, then there exists a continuous $f$ such that $M$ is an $\omega$ - limit set for $f$ and each limit point of $M$ is a fixed point of $f$. Recently Ceder [C] obtained some results concerning the disposition of fixed points in $\omega$ - limit sets.

We end with a few remarks and problems.

1. We now know that every nonempty nowhere dense compact set is an $\omega$ - limit set for some continuous $f$. The same is true for any finite union of closed intervals. (No other compact sets are realizable as $\omega$ - limit sets of continuous functions.)

We have also seen that not every such set is realizable as an $\omega$ - limit set for a differentiable function. This leads to the obvious problem.

Problem 1: Characterize those sets that are $\omega$ - limit sets for differentiable functions, for $C^1$ functions, or $C^\infty$ functions etc.

In the other direction we note that by increasing the class of functions to $\mathcal{DB}_1$, all nonempty closed subsets of $I$ are $\omega$ - limit sets [BCP]. (Thus, one can combine intervals with nowhere dense sets to obtain $\omega$ - limit sets for functions in $\mathcal{DB}_1$, but not for functions in $C$.) We mention that the class $\mathcal{DB}_1$ may appear more naturally in some contexts than the class $C$. For example, Newton's Method leads to iterations of functions of the form $x - \left(\frac{f(x)}{f'(x)}\right)$ for $f$ differentiable, but not necessarily continuously differentiable.

In [BCP] we show that a suitable modification of the function in our earlier example gives rise to a function $g \in \mathcal{DB}_1$ that has a homeomorphic copy of each nonempty nowhere dense closed subset of $I$ as an $\omega$ - limit set.

Problem 2 below asks for more.

Problem 2. Does there exist a function $f \in \mathcal{DB}_1$ such that to every nonempty compact set $M$ corresponds a homeomorphic copy $M_o \subset I$ and a point $x_o \in I$ such that $M_o = \omega_f(x_o)$?

We close by mentioning in this connection that there exists a Darboux measurable function $f$ that has every nonempty closed subset in an $\omega$ - limit sense. (Every, not just up to homeomorphism.)

## References


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