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LIMITS UNDER THE INTEGRAL SIGN[1]

Using a decomposable division space, we study

$$(1) \qquad \lim_{n\to\infty} \int_E f_n \, dm = \int_E \lim_{n\to\infty} f_n \, dm ,$$

where the f_n are functions of points with values in a space K, and m is a function of interval-point pairs with values in K or real or complex scalars, so that the values of f_n and m can be multiplied together. When K is linear with real scalars a and a norm $||k||$ satisfying $||ak|| = |a|\cdot||k||$, it is usual to have properties (i) $V(m;A;E) < \infty$, (ii) $||f_n-f|| \to 0$ m-almost everywhere in E, (iii) Fm and $F_n m$ $(n = 1,2, \ldots)$ integrable on E, and (iv) $m \geq 0$ and $||f_n|| \leq F$ $(n = 1,2, \ldots)$. These are a type of Arzela-Lebesgue condition in K. But (i) and (iv) restrict the test; (i) cuts out many applications to Feynman integration. Again, not all topological groups have even a group norm, while the restriction to $f_n(t)m(I,t)$ is a weakness. Generalizing to $h_n(I,t)$, we have a problem highlighted by the following examples.

On the real line, for each fixed integer $j \geq 2$ let $h_j([u,v),t) = v-u$ if $(j+1)u/j < v \leq ju/(j-1)$ $(u > 0)$, and otherwise let $h_j = 0$. Then $h = \sum_{j=2}^{\infty} h_j = v-u$ $(u < v \leq 2u)$ and otherwise $h = 0$, and the guage integrals $\int_{[0,1)} dh_j = 0$ (all $j \geq 2$), $\int_{[0,1)} dh = 1$, and $\sum_{j=2}^{\infty} h_{2j}$ is not integrable over any interval of $[0,1)$.

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These difficulties can be avoided by a slight change in the paper, "Generalized integrals of vector-valued functions," Proc. London Math. Soc. (3) 19 (1969), 509-536, MR 40 #4420, especially p.527. For $(K,\mathscr{G})$ a topological linear space let each $(I,t) \in U^1$ have a sequence $(Z^j(I,t))$ of sets of K containing the zero z, such that for each neighbourhood G of z, a positive integer j and a $U \in A|E$ exist and for all divisions $\mathcal{E}$ over E from U,

(2) $\quad (\mathcal{E})\ \Sigma\ Z^j(I,t) = \{\ (\mathcal{E})\ \Sigma\ z^j(I,t)\ :\ z^j(I,t) \in Z^j(I,t)\ \} \subseteq G.$

This is invariant under the action of real continuous linear functionals, and only differs from the paper in that $Z^j(I,t)$ replaces $Z^j(I)$.

For each integer n let $h_n(I,t)$ be integrable over E, let $X \subseteq T$, $U_+ \in A|E$, $U_+ \subseteq U$, and for each integer $j > 0$ and each $(I,t) \in U_+$ let $k(j;I,t)$ be an integer. For $h : U_+ \longrightarrow K$ let

(3) $\quad h_n(I,t) - h(I,t) \in Z^j(I,t)\ \ ((I,t) \in U_+,\ t \notin X,\ n \geq k(j;,I,t)),$

(4) $\quad h$ and h_n are of variation zero in X, relative to E,A.

For simplicity we can replace h, h_n by $h\cdot\chi(\backslash X;.)$, $h_n\cdot\chi(\backslash X;.)$, respectively, and can omit (4), taking X empty. This makes no difference to the integrability nor the value of the integral.

Theorem 1. *Let $(T,\mathcal{T},A)$ be a decomposable division space with K the real line or complex plane, let $h_n(I,t) = f_n(t)m(I,t)$, $h(I,t) = f(t)m(I,t)$ (all $(I,t) \in U$) and let X_1 be the set of t where $f_n(t) = f(t)$ for some integer $N(t)$ and all $n \geq N(t)$, and let the $Z^j(I,t)$, G be spheres $S(0,r)$ for various radii $r > 0$. Then (2), (3) imply that (a) the set X_2 of t where $f_n(t)$ fails to converge to $f(t)$, has m-variation zero, (b) $f_n(t)$ is bounded in n for each $t \in X_2$ for which there is an $(I,t) \in U_+$ with $m(I,t) \neq 0$, (c) m is VBG^* in $\backslash X_1$. Conversely, (a), (b), and (c) imply (2) and (3).*

It follows that for K the real line or complex plane and given the geometric constructions with $f_n m$, fm, (2) and (3) are equivalent to convergence m-almost everywhere and we are no further. The construction is of more value in more general spaces K.

Theorem 2. *In Theorem 1, h is integrable over E if and only if there are a compact set C of arbitrarily small diameter, some integer-valued M over U^1, some $U \in A|E$, and all divisions $\mathcal{E}$ of E from U, such that*

$$(\mathcal{E})\ \Sigma\, h_{n(I,t)}(I,t) \in C \quad (\text{all } n(I,t) \geq M(I,t)). \tag{5}$$

Theorem 3. *Given C and the conditions of Theorem 2 with (5), a necessary and sufficient condition for (1) is that there are an integer J, a compact set C_1 of arbitrarily small diameter, either containing a neighbourhood of the value of the integral over E of h, or such that $C \cap C_1$ is not empty, and a $U_n \in A|E$, with*

$$(\mathcal{E})\ \Sigma\, h_n(I,t) \in C_1 \tag{6}$$

for all divisions $\mathcal{E}$ of E from U_n with $n \geq J$.