Brian S. Thomson, Department of Mathematics and Statistics, Simon Fraser University, B.C., Canada V5A 1S6.

SYMMETRIC DERIVATIVES AND SYMMETRIC INTEGRALS *

In this talk I will speak about several symmetric derivatives, their related integrals, and the problem in trigonometric series which has motivated their study. In particular I wish to report on some work done jointly with David Preiss on the approximate symmetric integral that will appear in the Canadian Mathematics Journal.

1. Symmetric derivatives. From the family of symmetric derivatives we consider the following variants:

- (ordinary symmetric derivative)

$$\mathrm{SD}f(x) = \lim_{h\to 0} \frac{F(x+h) - F(x-h)}{2h}.$$

- (second order symmetric derivative)

$$\mathrm{SD}^2F(x) = \lim_{h\to 0} \frac{F(x+h) + F(x-h) - 2F(x)}{h^2}.$$

- (symmetric Borel derivative)

$$\mathrm{SBD}F(x) = \lim_{h\to 0} \frac{1}{h} \int_0^h \frac{F(x+t) - F(x-t)}{2t}\, dt$$

- (symmetric Cesàro derivative)

$$\mathrm{SCD}F(x) = \lim_{h\to 0} \frac{1}{h^2} \left\{ \int_x^{x+h} F(t)\, dt - \int_{x-h}^{x} F(t)\, dt \right\}$$

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- (approximate symmetric derivative)

$$\mathrm{ASD}f(x) = \mathrm{ap}\text{-}\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h}.$$

The symmetric derivative itself is pretty well known by now. I recently came across an elementary calculus text that introduced it before introducing the ordinary derivative. It was presented as a possible method of computing tangents. Several examples were computed ending with the computation of a "tangent" to the curve $y = |x|$ at the point $(0,0)$; since the symmetric derivative obliges with a tangent at a point at which, according to the author no tangent should exist, the symmetric derivative was dismissed and did not reappear in the text. Of course for us this "failure" of the symmetric derivative (it exists when a derivative should not) is an asset, not a liability.

These symmetric derivatives, while they might be motivated purely as technical generalizations of the derivative, and adding to our understanding of the structure of real functions, are most clearly exhibited as flowing from the expression

$$f(x) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[f(x+t) - f(x-t)]$$

which defines the *even* and the *odd* parts of the function f at the point x.

The derivative of the odd part of f at $t = 0$ is exactly the symmetric derivative SD $f(x)$ of the function f at the point x as expressed above. Differentiability of the even part of f at $t = 0$ is easily checked to be equivalent to the requirement that

$$\lim_{t \to 0} \frac{f(x+t) + f(x-t) - 2f(x)}{t} = 0$$

which condition is usually called the *smoothness* of the function f at the point x. This notion was first considered by Riemann in his famous memoir on trigonometric series which is where the second symmetric derivative was first introduced and used.

The connection of these ideas with trigonometric series is by now well known. If

$$a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

is the Fourier series of a function f then the convergence behaviour or the summability behaviour of the series at a point x or of the differentiated series (differentiated an even number of times) depends on properties of the function

$$t \to \frac{1}{2}[f(x+t)+f(x-t)-2f(x)]$$

near the point $t = 0$, that is to say on properties of the even part of f at x. For a well known example a theorem of Dini asserts that the Fourier series of a function f converges at a point x to the sum $f(x)$ provided that the integral

$$\int_0^\pi \frac{|f(x+t)+f(x-t)-2f(x)|}{t}\,dt$$

is finite.

In a similar way the behaviour of the conjugate Fourier series for f or of the differentiated series (differentiated an odd number of times) depends on the odd part of f at x, that is depends on properties of the function

$$t \to \frac{1}{2}[f(x+t)-f(x-t)]$$

near the point $t = 0$. Again, for example, the result analogous to that of Dini cited above has been given by Pringsheim and asserts that the conjugate Fourier series of a function f converges at a point x to the sum $f(x)$ provided that the integral

$$\int_0^\pi \frac{|f(x+t)-f(x-t)|}{t}\,dt$$

is finite.

2. Symmetric integrals. By a *symmetric integral* we mean an integral obtained from some kind of symmetric derivation process. We can give a brief summary of the kinds of integrals that have so far been introduced in this manner. If we ask first for a symmetric integral based on the ordinary symmetric derivative there are not too many instances. Certainly a Perron approach could be based on any of the following monotonicity theorems for the symmetric derivative: Khintchine (1927), Mukhopadyay (1966), Pu and Pu (1973), Kundu (1974), Weil (1976), Evans (1978), Larson (1983) and Freiling (1989). As far as I can tell there have been no Perron type integrals proposed based on the first order symmetric derivative. Such an integral would invert the symmetric derivatives of continuous functions and might be found useful in the study of trigonometric series. There are other approaches that have been used however. Denjoy (1955) indicated a symmetric totalization process for the inversion of symmetric derivatives of continuous functions. A symmetric integral as a limit of Riemanns sums has been also sketched in Henstock (1968) and in Kurzweil and Jarník (1987) although without much detail in either case. In this talk I would like to present an alternative to these last two integration procedures by basing such an integral on a different covering lemma than was used in those papers. This material has not appeared elsewhere and is intended largely as an introduction to the approximate symmetric integral.

There have been a number of symmetric integrals based, directly or indirectly, on the second order symmetric derivative. For a Perron approach see James (1951) and (1955), Marcinkiewicz and Zygmund (1936), Burkill (1951), and S. J. Taylor (1955). The James integral is directly based on the second order symmetric derivative and focuses on the problem of recovering (up to a linear function) a function G given its second order derivative $SD^2G(x)$ everywhere. This is rather well known since it is the integral that Zygmund chose to present in his treatise on trigonometric series with regard to the coefficient problem. Zygmund's own solution of that problem appeared much earlier in the paper of Marcinkiewicz and Zygmund just mentioned; although this is a "first order integral" and is based on the symmetric Borel derivative it can be considered to belong really to the second order symmetric derivative. The same can be said for the Burkill integral, based on the symmetric Cesáro derivative. Denjoy (1941) solved the same problem by a second order symmetric totalization process.

Finally let us mention those symmetric integrals that are based on the approximate symmetric derivative. Certainly for a Perron approach what is needed is any monotonicity theorem for the approximate symmetric derivative. The first try for such an integral is in Kubota (1971). This is unfortunately restricted by two difficulties: the monotonicity theorem used required approximately continuous functions and so restricted the applicability of the integral (in particular it does not help solve the coefficient problem for trigonometric series), and the proof of the monotonicity theorem itself was in error (see the review of H. Burkill MR47#2010). The other proofs of this monotonicity theorem (Mukhopadyay (1966) and Kundu (1973)) suffer from the same defect. Fortunately in Freiling and Rinne (1989) we have an apparently correct proof of this difficult theorem and so Kubota's Perron integral can be justified and developed in greater generality. Finally we indicate here that an approximate symmetric integral can be produced as a limit of Riemann sums (this will appear in the paper of Preiss and Thomson mentioned earlier).

3. Coefficient problem for trigonometric series. The following problem is the main motivation for the study of symmetric integrals and was certainly the primary motivation for our work on the approximate symmetric integral. Let the series

$$a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

converge everywhere to a function $f(x)$. Then these observations can be made:

- the coefficients are unique.
- if f is Lebesgue integrable then the series is necessarily the Fourier series for f.
- f need not be Lebesgue or even Denjoy-Perron integrable.

The problem then is "how may the coefficients a_n, b_n be determined from f"? Is there a more general integration procedure for which the series is a Fourier series for f? The problem can be varied by allowing an exceptional set of divergence or by replacing ordinary convergence by a summability method (in which case some growth condition on the coefficients needs to be placed to obtain uniqueness).

The remarks which follow are well known in the study of trigonometric series and illuminate the problem.

(1) Suppose that $\{b_n\}$ is a sequence of real numbers decreasing to 0 but with $\sum_{n=1}^{\infty} b_n/n = \infty$. Then the trigonometric series $\sum_{k=1}^{\infty} b_k \sin kx$ converges everywhere to a finite value $f(x)$, f is not integrable in the senses of Riemann, Lebesgue or Perron. However the integrated series

$$F(x) = \sum_{k=1}^{\infty} \frac{-b_k \cos kx}{k}$$

converges everywhere except at 0, $\pm 2\pi$, ... and $\mathrm{SD}F(x) = f(x)$ everywhere.

(2) Let the trigonometric series

$$a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

have coefficients satisfying the condition $\sum_{k=1}^{n} k\sqrt{a_k{}^2 + b_k{}^2} = o(n)$ as $n \to \infty$. Then the formally integrated series

$$F(x) = xa_0/2 + \sum_{k=0}^{\infty} (b_k \cos kx - a_k \sin kx)/k$$

converges everywhere to a continuous function $F(x)$ and $\mathrm{SD}F(x) = f(x)$ at every point x at which the series converges.

(3) Let the trigonometric series

$$f(x) = a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

converge everywhere. Then the twice formally integrated series

$$G(x) = x^2 a_0/4 - \sum_{k=0}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{k^2}$$

converges everywhere to a continuous function $G(x)$ and the second symmetric derivative of G recovers f, $\mathrm{SD}^2 F(x) = f(x)$.

(4) Let the trigonometric series

$$f(x) = a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

converge everywhere. Then the formally integrated series

$$F(x) = xa_0/2 + \sum_{k=0}^{\infty} (b_k \cos kx - a_k \sin kx)/k$$

converges almost everywhere and

$$\mathrm{SBD}F(x) = \ \mathrm{SCD}F(x) = \ \mathrm{ASD}F(x) = f(x)$$

everywhere.

From these remarks it is clear that the coefficient problem is intimately related to the various symmetric derivatives and that the corresponding symmetric integrals should provide some means of solving the problem. The remarks (3) and (4) are directly the observations that lead to the solution of the coefficient problem due to Denjoy, Marcinkiewicz and Zygmund, James and Burkill. It should be noted though that (3) directly provides (4) for the symmetric Cesáro derivative, and nearly directly for the symmetric Borel derivative. The statement for the approximate symmetric derivative is somewhat deeper.

4. Covering lemmas. A Riemann type integral based on the symmetric derivative is alluded to in Henstock (1968) and sketched out in Kurzweil and Jarník (1987). In their approach a function f defined on an interval $[a,b]$ is said to have a symmetric integral with

$$c = \int_a^b f(x)\,dx$$

provided that for every $\epsilon > 0$ there is a positive function δ defined on $[a,b]$ with the property that for any finite sequence

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

with the properties $x_1 - x_0 < \delta(a)$, $x_n - x_{n-1} < \delta(b)$, and for $k = 2, 3, \ldots n-1$,

$$x_k - x_{k-1} < \delta\left(\frac{x_k + x_{k-1}}{2}\right),$$

the expression

$$f(a)(x_1 - x_0) + \sum_{k=2}^{n-1} f\left(\frac{x_k + x_{k-1}}{2}\right)(x_k - x_{k-1}) + f(b)(x_n - x_{n-1})$$

differs from c by no more than ϵ. Of course the justification required for the integral is that for every such function δ partitions of this type exist.

Let us place this in a more useful geometric language, since the symmetric derivative can be characterized in terms of the geometry of symmetric covers. By a *symmetric cover* is meant a collection β of closed intervals with the property that for every x there is a $\delta(x) > 0$ so that

$$[x - h, x + h] \in \beta$$

for every $0 < h < \delta(x)$. Then the covering lemma that justifies the above integral is the following, attributable to McGrotty (1962).

Lemma 1 *If β is a symmetric cover then for every x there is a denumerable set $C_x \subset (x, \infty)$ so that β contains a partition of $[x - w, x + w]$ for every $x + w \notin C_x$.*

The key covering lemma that permits a more general Riemann type integral to be defined relative to the concept of symmetric covers has appeared in a note of Preiss and Thomson in this Exchange (1988).

Lemma 2 *Let β be a symmetric cover. Then there is a co-countable set B so that β contains a partition of every interval whose endpoints belong to B.*

Since there are exceptional countable sets that need to be handled at every turn it is natural to introduce a countable exceptional set into the notion of symmetric covers in a simple manner: by a *near-symmetric cover* is meant a collection β of closed intervals with the property that for some countable set C and every x there is a $\delta(x) > 0$ so that $[x - h, x + h] \in \beta$ for every $0 < h < \delta(x)$ provided that both $x - h$ and $x + h$ belong to $\mathbf{R} \setminus C$. Again one can prove the following extension.

Lemma 3 *Let β be a near-symmetric cover. Then there is a co-countable set B so that β contains a partition of every interval whose endpoints belong to B.*

It should be noted that in the definition of a near-symmetric cover a single exceptional countable set is permitted, not a different countable set at each point. We might have tried: a *co-countable-symmetric* cover is a collection β of closed intervals with the property that for every x there is a some countable set $C_x \subset (0, \infty)$ and a $\delta(x) > 0$ so that $[x - h, x + h] \in \beta$ for every $0 < h < \delta(x)$ with $h \notin C_x$. If we had tried to adopt the latter the proof of lemma 2 would not have succeeded. Indeed Sierpiński (1936) gives an example (under CH) of a non-measurable function f for which

$$\{y : f(x + y) \neq f(x - y)\}$$

is countable for each x. (On the other hand Professor C. Freiling has recently shown that, under the negation of CH, such a covering lemma would be available.)

For the same reason an exceptional density zero set cannot be used either! We might have tried: an approximate–symmetric cover is a collection β of closed intervals with the property that for every $x \in X$ there is a measurable set A_x of density 1 at 0 so that $[x-h, x+h] \in \beta$ for every $h \in A_x$. Again there would be no suitable covering lemma. To gain an acceptable covering lemma on which an approximate symmetric integral can be based we need the following.

A collection β is said to be a *measurable approximate symmetric cover* if there is a measurable set $T \subset \mathbf{R} \times (0, \infty)$ such that $[x-t, x+t] \in \beta$ whenever $(x,t) \in T$, and for every x

$$\limsup_{h \searrow 0} |\{t \in (0,h);\ (x,t) \notin T\}|/h = 0.$$

Then we can prove the covering lemma that justifies an approximate symmetric integral. This lies very much deeper than the preceding lemmas (as the measurability assumption alone might suggest).

Lemma 4 *Let β be a measurable approximate symmetric cover. Then there is a set B of full measure so that β contains a partition of every interval whose endpoints belong to B.*

5. The symmetric integral. Now we can present symmetric integrals based on these covering lemmas. The integral defined at the beginning of the previous section is not truly a "symmetric" integral, but rather some kind of a hybrid since it uses special conditions at the endpoints. We restrict ourselves to 2π–periodic functions as the presentation is simplest in this case.

Definition 5 Let f be a 2π–periodic function. We say that f has a *symmetric integral* if there is a number c such that for every $\epsilon > 0$ there is symmetric cover β such that, for any partition

$$x_0 < x_1 < \ldots < x_{n-1} < x_n = x_0 + 2\pi$$

that has all $[x_{i-1}, x_i] \in \beta$,

$$\left| \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1}) - c \right| < \epsilon.$$

Because such partitions exist this number c, if it exists, is unique and we may write it as $\int_0^{2\pi} f(x)\,dx$. More generally we also have a *near-symmetric* integral by using the near-symmetric covers instead, and an *approximate symmetric* integral by using measurable approximate symmetric covers. These integrals are in order of increasing generality.

6. Properties of the symmetric integrals. The following properties summarize some of the results that can be established for the symmetric integrals.

• Let f be a 2π-periodic function that is integrable in either the Riemann, Lebesgue or Perron senses. Then f has a symmetric integral and

$$\int_0^{2\pi} f(x)\,dx$$

has the same value in any of these senses.

• The symmetric integrals are incompatible with the integrals of James, Burkill and Marcinkiewicz–Zygmund. (One can exist when the other fails to exist and the values can differ even when both exist.)

• Let f be a 2π-periodic function with the property that SD $F(x) = f(x)$ nearly everywhere for some symmetrically continuous function F. Then f has a symmetric integral and

$$\int_0^{2\pi} f(x)\,dx = F(t + 2\pi) - F(t)$$

for nearly every t. If ASD $F(x) = f(x)$ nearly everywhere for some measurable, approximately symmetrically continuous function F then f has an approximate symmetric integral and

$$\int_0^{2\pi} f(x)\,dx = F(t + 2\pi) - F(t)$$

for almost every t.

• If a 2π-periodic function has a symmetric, near-symmetric or approximate symmetric integral then it is necessarily measurable.

• If a 2π-periodic function has a symmetric, near-symmetric or approximate symmetric integral and is nonegative then it must be Lebesgue integrable.

• Let f be a 2π-periodic function. Then f has a near-symmetric integral if and only if there is a co-countable set B and a function F defined on B

so that for every $\epsilon > 0$ there is a near–symmetric cover β that contains a partition of every interval with endpoints in B and such that for every $\gamma \subset \beta$ where γ is a finite collection of non-overlapping intervals

$$\sum_{[y,z]\in\gamma} \left| F(z) - F(y) - f\left(\frac{y+z}{2}\right)(z-y) \right| < \epsilon.$$

In this case

$$\int_0^{2\pi} f(x)\,dx = F(b+2\pi) - F(b)$$

for all $b \in B$.

• Let f be a 2π–periodic function. Then f has a near–symmetric integral if and only if there exists a function F defined on a co-countable set B such that

(i) F is near–symmetrically ACG_*, and

(ii) $\mathrm{D}_B F(x) = f(x)$ almost everywhere.

In that case f has a near–symmetric integral and

$$\int_0^{2\pi} f(x)\,dx = F(t+2\pi) - F(t)$$

for nearly every t.

In this last statement we say that a function F defined on a co-countable set is *near–symmetrically* ACG_* if for every set Z of measure zero and every $\epsilon > 0$ there is a near–symmetric cover β of the set Z such that

$$\sum_{i=1}^{n} |F(y_i) - F(x_i)| < \epsilon$$

for every sequence $\{[x_i, y_i]\} \subset \beta$ of non-overlapping intervals. It is not transparent that a function that is ACG_* in the usual sense is ACG_* in this sense but it can be shown as part of the usual development of the Henstock-Kurzweil integral.

A similar version for the approximate symmetric integral is available. For further details on this integral including an integration by parts formula and a Perron-type integral the paper of Preiss and Thomson must be consulted.

7. Trigonometric series. The symmetric integrals provide a solution to the coefficient problem in trigonometric series. The following can be proved:

Let the series

$$a_0/2 + \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$$

converge nearly everywhere to a function $f(x)$. Then f has an approximate symmetric integral and the series is a Fourier series for f (relative to this integral). If

1. $a_n = 0$ and $b_n \searrow 0$, or if
2. the series has coefficients satisfying the condition

$$\sum_{k=1}^{n} k\sqrt{a_k{}^2 + b_k{}^2} = o(n) \text{ as } n \to \infty$$

then the function f has a symmetric integral and the series is a Fourier series in this narrower sense.

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