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Lifting: the connection between functional representations of vector lattices

(summary)

The Ogasawara-Maeda [**LZ**] theorem states that every Archimedean vector lattice (= Riesz space) is representable as a subspace of the densely finite, continuous extended real-valued functions on some extremally disconnected compact Hausdorff topological space. I. Fleischer [**F2**] has recently given a new proof of this by passing through a representation of Carathéodory's place ("pointless") functions. These latter can also be realized as (co-meagrely finite) measurable functions modulo those which vanish outside of members of a σ-ideal via the Loomis-Sikorski theorem, from which Fleischer obtains another representation of the vector lattice. The use of a certain Stone space in both cases leads to the feeling that the representations must be "the same". We verify this, and give the connection with lifting theory of measure spaces.

Let $\mathcal{B}$ denote the σ-field of *Baire-property* (i.e., having a meagre symmetric difference with some open set) sets in an extremally diconnected (open sets have open closures) Baire space, or more generally, the σ-field of sets differing by a meagre from a cozero set in a basically disconnected [**GJ**] (cozero sets have open closures) in a basically Baire space (no nonvoid meagre cozero sets). Denote by

$\mathcal{L}^0$	the space of $\mathcal{B}$-measurable real functions
$\mathcal{N}_F$	its ideal of co-meagrely null functions
$\mathcal{L}^\infty$	its subspace of co-meagrely bounded functions
$\mathcal{C}$	the continuous bounded functions
$\mathcal{C}_\bullet$	the continuous densely finite extended real-valued functions.

THEOREM.

(1) *There is a lifting c of $\mathcal{B}$ modulo the meagre sets, with image the Boolean algebra of clopen sets.*

(2) *Integration with respect to $\lambda(A) = \mathbf{1}_{c(A)}$ is a strong lifting of $\mathcal{L}^\infty$ modulo $\mathcal{N}_F$, rendering $\mathcal{C}$ and $L^\infty = \mathcal{L}^\infty/\mathcal{N}_F$ isometrically isomorphic, and*

(3) *is a strong essential lifting of $\mathcal{L}^0(\mathcal{B})$ modulo its co-meagrely null functions, rendering $\mathcal{C}_\bullet$ and $L^0 = \mathcal{L}^0/\mathcal{N}_F$ isomorphic.*

Thus, c is an idempotent Boolean homomorphism on $\mathcal{B}$ onto its subalgebra of clopens with kernel the meagres; integration is an idempotent vector lattice (and ring) homomorphism on $\mathcal{L}^\infty$ selecting a (the) continuous bounded representative out of each class modulo the null functions,

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and is a homomorphism on $\mathcal{L}^0$ onto $C_\bullet$, sending each f onto a continuous function which differs from f only on a meagre set. (The word "strong" means $\int f\, d\lambda = f$, for continuous f.)

Application of (1) to the Stone space of a Boolean σ-algebra yields the Loomis-Sikorski Theorem [S,29.1] that every Boolean σ-algebra is isomorphic to a σ-field modulo a σ-ideal.

The family of bands (= order-closed ideals) in an Archimedean vector lattice V is a complete Boolean algebra; its Stone space is therefore a compact (hence Baire) extremally disconnected Hausdorff space and (3) shows that the two representations of V amount to the same thing.

In a nullset-complete finite measure space $(X, \mathcal{F}, \mu)$ (to take the simplest case) one can define the *lifting* topology by taking as a base for the open sets, the range of a lifting ℓ of $\mathcal{F}$ modulo $\mathcal{N}_\mu$, the μ-nullsets. This topology is extremally disconnected and Baire, though neither Hausdorff nor compact. The results above can be used to obtain liftings of $\mathcal{L}^\infty(\mu)$ and essential liftings of $\mathcal{L}^0(\mu)$ from liftings of the measurable sets.

Proofs and related results will appear in the the full version of this paper.

REFERENCES

[Fi] P.A. Fillmore, *On topology induced by measure*, Proc. Amer. Math. Soc. **17** (1966), 854–857.

[Fl] I. Fleischer, *Place functions, alias continuous functions on the Stone space*, Proc. Amer. Math. Soc. (to appear).

[F2] ———, *Functional representation of vector lattices*, Proc. Amer. Math. Soc (to appear).

[GJ] L. Gillman and M. Jerison, "Rings of Continuous Functions," Van Nostrand, Princeton, 1960.

[KP] J. Kupka and K. Prikry, *The measurability of uncountable unions*, Amer. Math. Monthly **91** (1984), 85–97.

[IT] A. and C. Ionescu Tulcea, "Topics in the Theory of Lifting," Springer-Verlag, Berlin 969.

[LZ] Luxemberg and A.C. Zanaan, "Riesz Spaces I," North Holland, Amsterdam, 1971.

[M] D. Maharam, *On a theorem of von Neumann*, Proc. Amer. Math. Soc. **9** (1958), 987–994.

[O] J. C. Oxtoby, "Measure and Category," Springer-Verlag, Berlin, 1971.

[S] R. Sikorski, "Boolean Algebras," Springer-Verlag, Berlin.

[T] T. Traynor, *An elementary proof of the lifting theorem*, Pacific J. Math. **53** (1974), 267–272.

[W] S. Willard, "General Topology," Addison-Wesley, Reading, 1970.