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BOLYAI-GERWIEN THEOREM AND HILBERT'S THIRD PROBLEM

This talk is meant to be a supplement of Paul Humke's lecture on Laczkovich's solution to the well-known circle-squaring problem of Tarski.

Laczkovich's solution states that a disc and a square with the same area are translation-equidecomposable. On the other hand, when the pieces in the decomposition are restricted to be polygons we are going to see that even two polygons with the area same are not necessarily translation-equidecomposable. To this end, let D denote the group of all the rigid motions of the Euclidean plane. For any subgroup G of D, two polygons A and B are said to be G-equidecomposable if there exist a positive integer n, n elements $g_1, g_2, \cdots, g_n \in G$, n non-overlapping polygons $A_1, A_2, A_3, \cdots, A_n$, and n non-overlapping polygons $B_1, B_2, B_3, \cdots, B_n$ such that
$A = \cup A_i$, $B = \cup B_i$ and $g_i(A_i) = B_i$ for $i = 1, 2, 3, \cdots, n$. Then the well-known Bolyai-Gerwien Theorem states that any two polygons of equal area are D-equidecomposable. Improving this result, Hadwiger and Glur have proved the following theorem, where S is the subgroup of D consisting of all the translations and all the center inversions.

Theorem 1. (Hadwiger-Glur). Any two polygons with the same area are S-equidecomposable.

Furthermore, using Hadwiger-Glur's work, Boltianskii has obtained the following result:

Theorem 2. Let G be a subgroup of D such that any two polygons with the same area are G-equidecomposable. Then $G \supset S$.

Therefore, we see that S is the smallest subgroup G of D such that every two polygons with the same area are G-equidecomposable.

We also know that Tarski's circle-squaring problem arises from the famous Banach-Tarski paradox, which states that any two bounded sets with non-empty interiors in the three-dimensional Euclidean space are equidecomposable. Contrast to this, when the pieces in the decomposition are restricted to be polyhedra we know that even two polyhedra with the same volume are not necessarily equidecomposable. This is just the Dehn's solution to the Hilbert's third problem, which says that the analogue of the Bolyai-Gerwien Theorem in the three-dimensional Euclidean space fails to hold. In fact, we have:

Theorem 3 (Dehn). A regular tetrahedron and a cube with the same volume are not equidecomposable.

To see this, Dehn first proved the following result:

Theorem 4 (Dehn). Let A, B be two equidecomposable polyhedra, and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_m$ be the radian measures of all the dihedral angles of A, and $\beta_1, \beta_2, \beta_3, \cdots, \beta_p$ be those of B. Then there exist integers $n_i > 0$, $q_j > 0$ such that $\sum_{i=1}^{m} n_i\alpha_i \equiv \sum_{j=1}^{p} q_j\beta_j \pmod{\pi}$.

The proof of Theorem 3 follows easily from Theorem 4 and the fact that $(\cos^{-1} \frac{1}{3})/\pi$ is irrational.

For all the results here and many more interesting results and unsolved problems, see the book: Hilbert's Third Problem by V. G. Boltianskii, V. H. Winston and Sons, Washington, D.C. (1978).