Paul D. Humke, Department of Mathematics, St. Olaf College, Northfield, MN 55057, humke@stolaf.edu

Remarks on Laczkovich's Circle-Squaring Proof

This paper, as advertised, contains none of my own work, but is an attempt to relay some of the flavor of the remarkable solution my friend Miklós Laczkovich has given to a 1925 problem of Alfred Tarski. As most of my knowledge of the history of this problem is directly attributable to Mik, I'll quote him rather liberally throughout this paper. I'll begin with the introductory remarks to his paper *Equidecomposability and discrepancy; a solution to Tarski's circle squaring problem* which will appear (or perhaps already has appeared) in Crelle's Journal.

Tarski's circle-squaring problem asks whether a disc is equidecomposable to a square; that is, whether a disc can be decomposed into finitely many parts which can be rearranged to obtain a partition of a square. The problem was motivated by the well known Banach-Tarski theorem stating that in $\mathbf{R}^3$ two sets are equidecomposable provided that they are bounded and have nonempty interior. In particular, any ball is equidecomposable to any cube. On the other hand, the existence of a Banach measure on the plane shows that a disc and a square can be equidecomposable only if they have the same area. (As for the proofs of these theorems, the further development of the theory and also for the history of Tarski's problem we refer to S. Wagon's book [*The Banach-Tarski paradox*, Cambridge Univ. Press, 1986]).

It is known that the answer to Tarski's problem is negative if we impose some restrictions either on the pieces of the decompositions or the isometries of the rearrangement. L. Dubins, M.H. Hirsch and J. Karush proved that the disc is not "scissor-congruent" to the square; that is, if the pieces are restriced to be Jordan domains (topological discs) then the disc and the square are not equidecomposable. The other result is due to R.J. Gardner.According to his theorem, the "circle-squaring" is impossible if the pieces are to be moved by isometries generating a locally discrete group (there are no restrictions on the pieces themselves).

In this paper we show that in spite of these negative results, the disc *is* equidecomposable to a square. Moreover, we only use translations in the rearrangement of the pieces.

Our proof is based on a sufficient condition for the equidecomposability of two bounded, measurable sets in terms of the discrepancy of certain special sequences (Theorem 5.1).

If T is a finite set, T is said to *decompose* two sets E_1 and E_2 if there is a bijection $\phi : E_1 -> E_2$ with $\phi(x) - x \epsilon T$ for each $x \epsilon E_1$; T *decomposes the intervals of length* d if T decomposes every pair of intervals of length d. The "parts" into which E_1 is decomposed are obtained by fixing tϵ T and considering those $e \epsilon E_1$ such that $\phi(e) - e = t$.

To understand some of the fundamental ideas governing the main proof, we follow Laczkovich and describe a one dimensional mathematical metaphor. The metaphor is this:

Find a set T which decomposes the intervals of length d for every $1/2 < d < 1$.

Fortunately, we are not without help.

THEOREM (1948 Hall-Rado) *The finite set T decomposes intervals I and J if and only if whenever $A \subset I$ and $B \subset J$ are finite, then*

1. $|(A + T) \cap J| \geq |A|$, *and*
2. $|(B - T) \cap I| \geq |B|$, *and*

TRY 1. Suppose T = $\{na : n = 0, \pm 1, ...\}$∩[0,1). In this case, if d is not in T it is rather easy to locate two intervals I and J of length d with $|I \cap T| > |J \cap T|$. This, however, violates the Hall-Rado conditions and we conclude that such T are not sufficiently rich for our purpose. In the next attempt, we use a slightly more sophisticated set, T.

TRY 2. Suppose T=$\{na + k : n, k = 0, \pm 1, ..., |n|, |k| \leq K,$ *a irrational*$\}$ For this investigation we need several definitions.

Definition 1. The *discrepancy* of a finite set S with respect to a second set H is

$$D(S,H) = |\frac{|S \cap H|}{|S|} - \lambda(H)$$

Definition 2. D(S) = sup{D(S,I) : I is an interval}

Definition 3. $\{x\}$ will denote the fractional part of x.

Now suppose NϵN and I, J are intervals. Define

$$S_N = \{\{na\} : 0 \leq n < N\}$$

and set $A_N = S_N \cap I$ and $B_N = S_N \cap J$. Then

$$A_N| - |B_N|| = ||S_N \cap I| - |S_N \cap J|| =$$

$$N|[\frac{|S_N \cap I|}{N} - \lambda(I)] - [\frac{|S_N \cap J|}{N} - \lambda(J)]|$$

$$N|D(S_N, I) - D(S_N, J)|.$$

If T decomposes I and J we can use the Hall-Rado result to give a second estimate.

$$A_N| = |S_N \cap I| \leq |(A_N + T) \cap J|$$

$$\leq |S_N \cap J| + 2K = |B_N| + 2K.$$

That is,

$$A_N| - |B_N|| \leq 2K.$$

It follows that

$$(*) \qquad |D(S_N, I) - D(S_N, J)| \leq \frac{2K}{N}$$

We again defer to Mik.

Now, one of the basic facts of the discrepancy theory is that the sequence of numbers $ND(S_N)$ cannot be bounded. This implies that we can choose a large N and an interval $I \subset [0,1)$ such that $ND(S_N; I) > 2K + 2$. We may also choose I with $\lambda_1(I) \geq 1/2$. On the other hand, it is easy to see that, among the intervals of any given length, there must be one satisfying $D(S_N; J) \leq 2/N$. Hence, there is an interval J such that $\lambda_1(I) = \lambda_1(J)$ and $(*)$ is not satisfied.

TRY 3. Suppose T $= \{k + na + mb\ : k, n, m = 0, \pm 1, ..., |k|, |n|, |m| \leq K\}$. Could such a set serve as the solution to our one dimensional problem? Analyzing as above we obtain that there is a constant C such that for each Nϵ**N**,

$$N^2 |D(S_N^*, I) - D(S_N^*, J)| \leq CN.$$

Here, $S_N^* = \{\{na + mb\} : 0 \leq n, m \leq N\}$. Now, $N^2 D(S_N^*, J)$ can be as small as "2" so that there are restrictions on the magnitude of $N^2 D(S_N^*)$. These restrictions, however, do not contradict the fact that $N^2 D(S_N^*)$ is unbounded; it could grow logarithmically for example. Such a set, T, then is a candidate solution to our one dimensional metaphor and indeed, Laczkovich affirms an appropriate candidate in his THEOREM 5.1. To continue we need three additional definitions.

Definition 4. $< x > = \min\{\{x\}, 1 - \{x\}\}$.

Definition 5. If $\mathbf{x}=(x_1, x_2)\epsilon\mathbf{R}^2$, then $(\mathbf{x})=(\{x_1\}, \{x_2\})$.

Definition 6. $S_N(\mathbf{u},\mathbf{x},\mathbf{y}) = \{(\mathbf{u} + n\mathbf{x} + \mathbf{y}) : 0 \leq n, k \leq N\}$.

With this background, let me make two absolutely obvious observations.

Observation 1. *Let Ψ be a nonnegative function such that $\Psi(n) < \log n$ and let H_1 and H_2 be measurable subsets of $[0,1]^2$ with $\lambda_2(H_1) = \lambda_2(H_2) > 0$. Suppose* $\mathbf{x}_o, \mathbf{y}_o, \mathbf{i}, \mathbf{j}\epsilon\mathbf{R}^2$ *are linearly independent over the rationals and*

$$N^2\ D(S_N(\mathbf{u},\mathbf{x_o},\mathbf{y_o}),H_i) \le \Psi(N)$$

for every $N\,\epsilon\mathbf{N}$, *i=1,2,* $\mathbf{u}\epsilon\mathbf{R}^2$. *Then there exists a bijective*

$$\phi : S_1(\mathbf{u}) \to S_2(\mathbf{u})$$

where $S_i(\mathbf{u}) = \{(n,k) : \mathbf{u} + n\mathbf{x_o} + k\mathbf{y_o}\ \epsilon\ H_i\}\quad i = 1,2.$

(As both $S_1(\mathbf{u})$ and $S_2(\mathbf{u})$ are countably infinite, there is really nothing to this.)

Observation 2.*Suppose the hypotheses of Observation 1 hold and let G be the group generated by* $\mathbf{x_o},\mathbf{y_o},\mathbf{i}$, *and* $\mathbf{j}$.

Then define

$$\mathbf{z_1} \sim \mathbf{z_2}\ \ iff\ \ \mathbf{z_1} - \mathbf{z_2}\epsilon G.$$

Let E be an equivalence class under $\sim$ *and suppose* $\mathbf{u}\epsilon E$ *If* $\mathbf{z}\epsilon E$ *there are unique interger coeficients n,m,k,p with*

$$\mathbf{z} = \mathbf{u} + n\mathbf{x_o} + m\mathbf{y_o} + k\mathbf{i} + p\mathbf{j}$$

If, in addition, $\mathbf{z}\epsilon H_1$, *then* $(\mathbf{u} + n\mathbf{x_o} + m\mathbf{y_o})\epsilon H_1$ *so that* $(n,m)\epsilon S_1(\mathbf{u})$. *It follows that* $\phi(n,m) = (n',m')\epsilon S_2(\mathbf{u})$ *and so there are* k' *and* p' *such that*

$$\mathbf{z}' = \mathbf{u} + n'\mathbf{x_o} + m'\mathbf{y_o} + k'\mathbf{i} + p'\mathbf{j}\epsilon H_2$$

It is now easy to see that $F(\mathbf{z}) = \mathbf{z}'$ is a bijection from $E \cap H_1$ to $E \cap H_2$ such that $F(\mathbf{z}) - \mathbf{z} \epsilon G$ for each $\mathbf{z} \epsilon E \cap H_1$. As E is arbitrary, F is a bijection from H_1 to H_2. That is, G decomposes H_1 and H_2.

Note that as G is not finite this use of "decompose" is technically incorrect. Unfortunately, G is countably infinite making Observation 2 a triviality. However, if $\phi(\mathbf{z}) - \mathbf{z}$ were bounded *then a finite subset of G would decompose H_1 and H_2 because $|F(\mathbf{z}) - \mathbf{z}| \leq 2^{1/2}$ for each $\mathbf{z} \epsilon H_1$. It is this property of the mapping $\phi : S_1(\mathbf{u}) \to S_2(\mathbf{u})$ which is critical in the proof of THEOREM 5.1 stated below.*

THEOREM 5.1 *Let Ψ be a nonnegative function such that*

$$\sum_{k=1}^{\infty} \frac{\Psi(2^k)}{2^k} < \infty.$$

Let H_1 and H_2 be measurable subsets of $[0,1]^2$ with $\lambda_2(H_1) = \lambda_2(H_2) > 0$. Suppose x_o, y_o, i, and j are linearly independent over the rationals and that

$$N^2 D(S_N(\mathbf{u}, \mathbf{x_o}, \mathbf{y_o}), H_i) < \Psi(N)$$

for every $N \epsilon \mathrm{N}, \mathbf{u} \epsilon \mathrm{R}^2, i = 1, 2$. Then H_1 and H_2 are (translation) equidecomposable.

Remark. The question remains: given such sets H_1 and H_2, where does one find the appropriate $\mathbf{x_o}$ and $\mathbf{y_o}$? The answer comes via an existence argument which uses techniques not available until the midsixties.

Using techniques of W. Schmidt one can prove the following estimates.

THEOREM 6.2 *For a.e. $(x,y) \epsilon \mathrm{R}^2$ and for every $\varepsilon > 0$ there is a $C > 0$ such that*

$$\sum_{k=1}^{n} \frac{1}{< kx >< ky >} \leq Cn log^{3+\varepsilon}(n)$$

$$\sum_{k=1}^{n} \frac{1}{k<kx><ky>} \leq Clog^{3+\varepsilon}(n)$$

$$\sum_{k=1}^{n} \frac{1}{k^2<kx><ky>} \leq \frac{Clog^{3+\varepsilon}(n)}{n}$$

The one dimensional version of the Erdős-Turán Theorem gives the following estimate for every $m\epsilon$ **N**.

$$D(S_N(u,x,y)) \leq \frac{6}{m} + \sum_{h=1}^{m} \frac{1}{h}|\frac{1}{N^2} \sum_{0\leq n,k\leq N} e^{2\pi ih\{u+nx+ky\}}$$

But then,

$$\sum_{0\leq n,k\leq N} e^{2\pi ih\{u+nx+ky\}}| = |\sum_{0\leq n,k\leq N} e^{2\pi ih(nx+ky)}| =$$

$$\frac{|e^{2\pi ihNx}-1||e^{2\pi ihNy}-1|}{|e^{2\pi ihx}-1||e^{2\pi ihy}-1|} \leq$$

$$\frac{1}{|sin(\pi hx)||sin(\pi hy)|} \leq \frac{4}{<hx><hy>}.$$

We then take x and y as in THEOREM 6.2 and let m=N to obtain:

THEOREM 6.3 *For a.e. $x,y\epsilon$*R *and for every $\varepsilon>0$, there is a $C>0$ such that*

$$D(S_N(u,x,y)) \leq \frac{Clog^{3+\epsilon}(N)}{N^2}$$

*for every $u\epsilon$*R *and every $N\epsilon$*N.

he two dimensional version of this theorem is as follows. In the next result, $D(S_N(\mathbf{u},\mathbf{x},\mathbf{y}))$ denotes the $\sup\{D(S_N(\mathbf{u},\mathbf{x},\mathbf{y}),I) : I$ is a two dimensional rectangle$\}$

THEOREM 8.4 *For a.e.* $\mathbf{x},\mathbf{y}\epsilon\mathbf{R}^2$ *and for every $\varepsilon>0$, there is a $C>0$ such that*

$$D(S_N(\mathbf{u},\mathbf{x},\mathbf{y})) \leq \frac{Clog^{6+\epsilon}(N)}{N^2}$$

for every $\mathbf{u}\epsilon\mathbf{R}^2$ *and every* $N\,\epsilon\mathbf{N}$.

What remains (other than the proof!) is to compute discrepancies of nonrectangular regions for sequences $S_N(\mathbf{u},\mathbf{x},\mathbf{y})$ and then to apply THEOREM 5.1. The required estimates use Fourier series for perturbations of discrepancies similar to that used to prove the Erdős-Turán Theorem. In the next theorem, H_f denotes the region under the graph of f.

THEOREM 9.1 *Let f be twice differentiable on* $[0,1]$, *f(0)=0,f(1)=1, and suppose there exist* $a,b,c,d > 0$ *such that for every* $x\epsilon[0,1]$

$$a \leq |f'(x)| \leq b \qquad\qquad c \leq |f''(x)| \leq d.$$

Then, for a.e. $\mathbf{x},\mathbf{y}\epsilon\mathbf{R}^2$ *and for every* $\varepsilon > 0$, *there is a* $C > 0$ *with*

$$D(S_N(\mathbf{u},\mathbf{x},\mathbf{y}),H_f) \ \leq \ \frac{Clog^{6+\varepsilon}(N)}{N^{3/2}}$$

for every $\mathbf{u}\epsilon\mathbf{R}^2$ *and each* $N\epsilon\mathbf{N}$.

Finally, let C be a circle. To prove the circle-squaring theorem it is sufficient to show that a wedge of C with central angle $\pi/4$ is translationally equidecomposable to a square having the same area. It is easy to see, however, that if A is an affine transformation then H_1 and H_2 are equidecomposable if and only if $A(H_1)$ and $A(H_2)$ are equivalent. By transforming the wedge with the appropriate $\pi/4$-shear, it is enough to show that the area beneath the transformed circular arc is equidecomposable to a square of the same area. By THEOREM 5.1, it is sufficient to find $\mathbf{x}_o,\mathbf{y}_o$ such that the associated discrepancies grow sufficiently slowly. The existence of these points is then guaranteed by THEOREMS 8.4 and 9.1.

It was neither my intent to prove the circle-squaring theorem here nor to even outline the proof in any detail. Rather, my pupose was to give a flavor of the proof and a few of the ideas it contains. The proof itself is even more remarkable than the result and if I have whetted your appetite for the real thing, this short note will have found it's mark. Bon appetit!