Richard G. Gibson, Department of Mathematics and Computer Science, Columbus College, Columbus, Georgia 31993.

Concerning Two Properties of Connectivity Functions

Let X and Y be topological spaces and let $f:X \rightarrow Y$. Then:

D. : f is a <u>Darboux function</u> if $f(C)$ is connected whenever C is connected in X.

Conn. : f is a <u>connectivity function</u> if the graph of f restricted to C, denoted by $f|C$, is connected in $X \times Y$ whenever C is connected in X.

A.C. : f is an <u>almost continuous function</u> if $U \subset X \times Y$ is any open set containing the graph of f, then U contains the graph of a continuous function $g:X \rightarrow Y$.

Ext. : f is an <u>extendable function</u> if there exists a connectivity function $g:X \times [0,1] \rightarrow Y$ such that $f(x) = g(x,0)$ for each x in X.

Let $f:[a,b] \rightarrow R$ be a function. Then:

P.R. : f has a <u>perfect road</u> if for each x in $[a,b]$ there exists a perfect set P having x as a bilateral limit point such that $f|P$ is continuous at x. If x is an endpoint, then the bilateral condition is replaced with a unilateral condition.

For real-valued functions defined on an interval $[a,b]$ we have only the following implications among the classes of functions defined above.

$$\text{Ext.} \rightarrow \text{A.C.} \rightarrow \text{Conn.} \rightarrow \text{D.}$$
$$\text{Ext.} \rightarrow \text{P.R.}$$

Question 1. Does A.C. + P.R. $\rightarrow$ Ext.?

Question 2. Does Conn. + P.R. $\rightarrow$ A.C.?

Question 3. Does D. + P.R. $\rightarrow$ Conn.?

The following is a discussion of these questions. Let $I = [0,1]$ and let $I^2 = [0,1] \times [0,1]$.

If A is any set and p is any point of A, then by a <u>quasi-component</u> of A containing p is meant the set consisting of p together with all points x of A such that A is not separated between p and x; i.e., there exists no separation $A = A_p \cup A_x$ where $\bar{A}_p \cap A_x = A_p \cap \bar{A}_x = \emptyset$, p is in A_p, and x is in A_x. For reference see [5]. The quasi-components of any set A are disjoint and closed in A. A quasi-component of A may not be connected but each component of A is contained in a single quasi-component. In general the quasi-components of a set may be different from the components of this set. However for a particular set associated with a connectivity function they are the same.

Theorem 1. If $g: I^2 \rightarrow I$ is a connectivity function, then the quasi-components of $I^2 - g^{-1}(z)$ are connected for each z in I.

The proof of this theorem is contained in the proof of theorem 2.1 of the paper by Hunt [2].

Corollary 1. If $g:I^2 \rightarrow I$ is a connectivity function and z separates $g(I^2) \subset I$, then any point of $g^{-1}([0,z))$ and any point of $g^{-1}((z,1])$ lie in different quasi-components of $I^2 - g^{-1}(z)$.

Corollary 2. If $g:I^2 \rightarrow I$ is a connectivity function and z separates $g(I^2) \subset I$, then $g^{-1}(z)$ separates I^2.

Corollary 2 is a generalization of a well-known fact that says that if $g:I^2 \rightarrow I$ is continuous and z separates $g(I^2) \subset I$, then $g^{-1}(z)$ separates I^2. The following example shows that the conclusion of corollary 1 is not true for Darboux functions and for almost continuous functions.

Example. Define $h:[-1,1] \times [0,1] \rightarrow [-1,1]$ by

$$h(x,y) = \sin(1/y) \text{ , if } y > 0 \text{ and}$$
$$h(x,0) = x \text{ otherwise.}$$

Now h is a Darboux function and an almost continuous function but not a connectivity function.

If $f:X \rightarrow Y$ is a function and B is a subset of Y, then a _leaf_ L of $f^{-1}(B)$ means that there exists a y in B such that L is a component of $f^{-1}(y)$.

Theorem 2. Let $g:I^2 \rightarrow I$ be a connectivity function and let $\varepsilon > 0$. If A is the union of all leaves of $g^{-1}(I)$ which have diameter greater than or equal to ε, then A is closed and the restriction of g to A is continuous.

Let $f:[a,b] \to R$ be a function. Then

SCIVP.: f has the <u>SCIVP</u> if for p and q in $[a,b]$ such that $p \neq q$ and $f(p) \neq f(q)$ and for any Cantor set K between $f(p)$ and $f(q)$ there exists a Cantor set C between p and q such that $f(C) \subset K$ and $f|C$ is continuous.

Using <u>theorem 1</u> and <u>theorem 2</u> we proved the following.

Theorem 3. If $g:I^2 \to I$ is an extension of $f:I \to I$ and g is a connectivity function, then f has the SCIVP.

However, there exists a function that is both A.C. and P.R which does not have the SCIVP. Thus the answer to <u>question 1</u> is no. These results will appear in a paper under preparation by H. Rosen, F. Roush, and me.

F. B. Jones and E. S. Thomas, Jr. [3] constructed a connectivity function $f:I \to I$ with its graph a G_δ -set that is not an almost continuous function. In [1] I defined <u>property B</u> and proved that for Darboux functions $[a,b] \to R$, property B and P.R. are equivalent. Recently H. Rosen [4] proved that for Darboux functions with its graph a G_δ -set, the function has property B and hence P.R. Thus the function constructed by Jones and Thomas is a connectivity function with a P.R. that is not almost continuous. So the answer to <u>question 2</u> is no.

The answer to <u>question 3</u> is unknown. But we have another question.

Question 4. Does there exist a Darboux function with its graph a G_δ -set that is not a connectivity function?

REFERENCES

1. R.G. Gibson, Darboux functions with a perfect road, under preparation.

2. J.H.V. Hunt, A connectivity map $f:S^n \to S^{n-1}$ does not commute with the antipodal map, Sociedad Mathmatica Mexicana Bolatin S.2, 16(1971), pp 43-45.

3. F.B. Jones and E.S. Thomas, Jr., Connected G_δ graphs, Duke Math. Journal, 33(1966), pp 341-345.

4. H. Rosen, G_δ Darboux functions, under preparation.

5. G.T. Whyburn, Analytic Topolgy, AMS Colloqium Publications 28 (1942).