Krzysztof Ciesielski, Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506
Lee Larson and Krzysztof Ostaszewski, Department of Mathematics, University of Louisville, Louisville, Kentucky 40292
Presented by K. Ciesielski

I-density Continuous Functions

Here, and in what follows I will stand for the ideal of first category subsets of R.

Definition of an I-density point of a set $A \subseteq R$.

Motivation: 0 is a density point of A iff

$$\lim_{n\to\infty} \frac{m\left[A\cap\left(-\frac{1}{n}, \frac{1}{n}\right)\right]}{\frac{2}{n}} = 1 \quad \text{iff} \quad \lim_{n\to\infty} m\left[n\cdot A\cap(-1,1)\right] = 2 \quad \text{iff}$$

$$\chi_{n\cdot A\cap(-1,1)} \underset{n\to\infty}{\to} \chi_{(-1,1)} \text{ in measure iff}$$

$$\forall (n_m) \exists (n_{m_k}) \lim_{k\to\infty} \chi_{n_{m_k}\cdot A\cap(-1,1)} = \chi_{(-1,1)} \quad I_m\text{-a.e.}$$

Definition (Wilczyński)

(1) 0 is an I-density point of A iff

$$\forall (n_m) \exists (n_{m_k}) \lim_{k\to\infty} \chi_{n_{m_k}\cdot A\cap(-1,1)} = \chi_{(-1,1)} \quad I\text{-a.e.}$$

(2) x is an I-density point of A iff
0 is an I-density point of $(-x)+A$

Let τ_I be a family of all Borel sets $A\subset R$ such that every point $x\in A$ is an I-density point of A.

Fact 1. τ_I is a topology on R.

Def. (1) τ_I is called I-density topology on R.

(2) A function $f:R\to R$ is said to be I-density continuous if it is continuous when domain and range are equipped with I-density topology, i.e., when $f:(R,\tau_I)\to(R,\tau_I)$ is continuous.

Remark. Unfortunately (R,τ_I) is not regular. To correct this another definition has been introduced.

Def. x is a deep I-density point of $A\subset R$ provided there exists a closed set $P\subset A$ such that x is on I-density point of P.

Similarly as before we define a deep I-density topology τ_{dI} and deep I-density continuous functions (as continuous $f:(R,\tau_{dI})\to(R,\tau_{dI})$).

Fact A. (R,τ_{dI}) is completely regular.

Fact B. A homeomorphism $h:R\to R$ (or $h:(a,b)\to(c,d)$) is I-density continuous iff it is deep I-density continuous.

Fact C. Ordinary topology $\subset \tau_{dI} \subset \tau_I$, ordinary topology $\not\subset$ density topology but density topology $\not\subset \tau_I$, $\tau_{dI} \not\subset$ density topology.

Results:

Theorem 0. If $h:R\to R$ and h^{-1} satisfy a local Lipschitz condition then h and h^{-1} are I-density continuous.

Theorem 1. Analytic functions are I-density continuous.

Example 1. There is a C^∞ homeomorphism that is not I-density continuous.

Example 2. There is a convex function on R that is not I-density continuous.

Corollary 1. There is a density continuous homeomorphism that is not I-density continuous.

Theorem 2. If f is I-density continuous then f is Baire *1, i.e., for every perfect set P there is a portion $Q=P\cap(a,b)\neq\phi$ such that $f|Q$ is continuous in the ordinary sense.

Theorem 3. The space of I-density continuous functions considered with the uniform convergence topology is 1-st category in itself.

Theorem 4. The space of continuous, I-density continuous functions is nowhere dense in the space C(R) of continuous functions.

Theorem 5. $\{f^{-1}(0) : f:R\to R$ is I-density continuous$\} = \{A\subseteq R$: A is $F\delta, G\delta$ and is I-density closed$\}$

Notation: C_I stands for $\circ$ semigroups of I-density continuous functions.

Theorem 6. The semigroup $(C_I, \circ)$ has inner automorphism property (i.e., every automorphism $\Phi: C_I\to C_I$ can be represented as $\Phi(f)=h\circ f\circ h^{-1}$ for some $h\in C_I$) but (R, τ_I) is not generated (i.e., the family $\{R\setminus f^{-1}(x): x\in R, f\in C_I\}$ does not form a subbase for (R, τ_I)).