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CONCERNING THE BAIRE CLASS OF TRANSFORMATIONS ON PRODUCT SPACES

In this note we solve a problem stated at the end of [6] and some related queries quoted in [5]. In what follows (X,d_X), (Y,d_Y) and (Z,d_Z) denote three complete, separable, metric spaces. If $f : X \times Y \to Z$, we shall call the family of transformations $f_x : Y \to Z$, $x \in X$ defined by $f_x(y) := f(x,y)$ the X-sections of f. The Y-sections are defined similarly by $f^y(x) := f(x,y)$. Numerous papers were devoted to conditions guaranteeing Borel measurability of a transformation expressed in terms of its sectionwise properties. (See a chart in [7], p. 169.) In particular [6] contains the following definition and theorem:

DEFINITION 0: ([6], df. 1.) A family $F \subset Z^X$ of transformations $f : X \to Z$ fulfills the property A_2 if for each nonvoid closed subset K of X there is a point $x^o \in K$ such that the family of restrictions $\{f|_K : f \in F\}$ is equicontinuous at x^o.

Recall that a family $G \subset Z^K$ is equicontinuous at x^o if for each number $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Z(g(x),g(x^o)) < \varepsilon$, whenever $x \in K \cap B(x^o,\delta)$ and $g \in G$ where $B(x^o,\delta)$ denotes the open ball in X centered at x^o with radius δ. Notice however that the above notion depends only on the topology of X and the uniformity of Z and thus the original metrics may be replaced by uniformly equivalent ones. Note also that in compliance with the terminology of B. Ricceri ([11], df. 13) the family F of definition 0 is composed of functions equi-belonging to the first Baire class. In case F consists of a single function f, the property A_2 simply means that f is Baire one.

THEOREM 0. ([6], th. 6 and remark 1 on p. 123.) If $g : X \times Y \to Z$ is a transformation for which all X-sections belong to Baire class α, $0 < \alpha < \Omega$, and all Y-sections fulfill property A_2, then g also belongs to Baire class α on $X \times Y$.

We need some concepts from [1]. A subset $E(x) \subset X$ is called a path leading to x if $x \in E(x)$ and x is an accumulation point of $E(x)$. A system of paths $E : X \to 2^X$ is said to satisfy the essential radius condition if for each positive function s on X there is a positive function r on X such that if $d_X(a,b) < \min\{r(a),r(b)\}$, then $E^s(a) \cap E^s(b) \neq \emptyset$ where $E^s(c)$ denotes $E(c) \cap B(c,s(c))$. (See [9], Ex. 9.A.3..)

The following definition is a simultaneous generalization of [4], df. 8 on p. 19 and [1], df. 5.1 on p. 110:

DEFINITION 1. Let $E : X \to 2^X$ be a system of paths. A family $F \subset Z^X$ is said to be E-equicontinuous at $x^o \in X$ if the family of restrictions $\{f|_{E(x^o)} : f \in F\}$ is equicontinuous at x^o in the sense mentioned after definition 0. If F is everywhere E-equicontinuous, then we say briefly that F is E-equicontinuous.

Note that in general there is no topology U on X for which E-equicontinuous transformations were exactly U-equicontinuous. In certain cases this happens, e.g. the density topology leads to the notion of approximate equicontinuity defined in [4]. On the other hand there is no topology T on X for which preponderantly continuous transformations [10] are exactly T-continuous and thus the notion of a preponderantly equicontinuous family, as defined in [5], p. 22 cannot be expressed in terms of any topology. We are now in a position to state the following:

PROPOSITION 1. Let $E : X \to 2^X$ be a system of paths satisfying the essential radius condition. Then any E-equicontinuous family $F \subset Z^X$ has property A_2.

Proof: We may assume that the space Z is bounded. Let $B(F,Z)$ denote the metric space of all mappings from F into Z endowed with the uniform metric $D(g_1,g_2) := \sup\{d(g_1(f),g_2(f)) : f \in F\}$. Let $h : X \to B(F,Z)$ be defined by $h(x)(f) = f(x) \in Z$; $x \in X$. Since F is E-equicontinuous, $h|_{E(x)}$ is continuous at x for every $x \in X$. It can be readily verified by using the methods developed in [1]. Theorem 5.2 on p. 110 and [12], Theorem 33.1 on p. 74 that this implies that for every nonempty closed set $K \subset X$, the restriction $h|_K$ has a point of continuity. (Although [1] and [12] only state this for real-valued functions defined on the real line, and use a different intersection condition, the proofs are valid, under suitable changes, for maps of X into arbitrary metric spaces.) Now this statement is obviously equivalent to the A_2 property of the family F and the proof is complete.

COROLLARY 1. Let $X = \mathbb{R}^n$ and let F be a preponderantly (in particular approximately) equicontinuous family of transformations with respect to the ordinary differentiation base. Then F has property A_2.

This follows from the fact that the corresponding systems of paths satisfy the essential radius condition providing a positive answer to the problem on p. 125 in [6] and, at the same time, to question 11 b), c) from [5].

Combining Theorem 0 and Proposition 1 we obtain a positive solution to question 11 d), e) from [5]:

COROLLARY 2. Let $X = \mathbb{R}^n$ and $f : X \times Y \to Z$ be a transformation for which all Y-sections create a ordinarily preponderantly (resp. approximately) equicontinuous family and all X-sections belong to Baire class α, $0 < \alpha < \Omega$. Then f belongs to Baire class α.

However, in the Corollary 2 the space X may be essentially generalized as in [4], p. 7-8.

REMARK 0. In the one dimensional case a system of paths of (ζ,λ)-density type with $\{\zeta,\lambda\} > 2^{-1}$ satisfies our intersection condition (cf. [1], th. 3.5). Thus transformations that are preponderantly equicontinuous even on just one side must have the A_2 property.

REMARK 1. The first sentence of the proof of Proposition 1 shows that the assumption concerning uniform boundedness in theorem 7 from [6] is superfluous, solving in that manner the remaining question 11 a) from [5].

REMARK 2. Let $X = \mathbb{R}^n$ and let $E : X \to 2^X$ be a system of paths such that x is an ordinary I-density point of $E(x)$ in the sense of [14], where I denotes the ideal of meager sets. Then E satisfies the essential radius condition as follows from [15], Theorem 3, and from the fact that in the case of $X = \mathbb{R}^n$ the star-porosity topology defined in [15] coincides with the I-density topology. In view of Remark 2, all Y-sections of an f in Corollaries 1, 2 may be even I-approximately equicontinuous.

DEFINITION 2. (See [9].) A transformation $f : X \to Z$ is said to be nonalternating if, whenever C is connected in Z, $f^{-1}(C)$ is connected in X. Sometimes such transformations are called inverse connected.

In case $X = Z = \mathbb{R}$ Definition 2 reduces to f being weakly increasing or decreasing. In the sequel we shall assume additionally that the space Z is a Banach space endowed with $d := \min\{1, d_Z\}$, where d_Z is the distance function induced by the norm.

PROPOSITION 2. Let $f : \mathbb{R} \times Y \to Z$ be a transformation whose Y-sections are nonalternating and all X-sections create a separable subspace of the space $B_1(Y,Z)$ of Baire 1 transformations. Then f is also of the first Baire class.

Proof: Let us put $h(x) := f_x \in B_1(Y,Z)$. We prove that h is a Baire 1 transformation. Since $h(\mathbb{R})$ is separable, each open set in this target space is a countable union of open balls. On the other hand each open ball $B(g,r)$ is a countable union of the closed balls $\bar{B}(g, r - 2^{-n})$, $n \in \mathbb{N}$. Therefore it suffices to prove that $h^{-1}(\bar{B}(g, r - 2^{-n}))$ are F_σ subsets of $\mathbb{R}$. Indeed, we have $h^{-1}(\bar{B}(g,s)) = \{x \in X : d(h(x),g) \leq s\} = \{x \in X : d(f(x,y),g(y)) \leq s$ for each $y \in Y\} = \bigcap_{y \in Y} (f^y)^{-1}(\{z \in Z : d(z,g(y)) \leq s\})$. All the balls $\bar{B}(g(y),s) \subset Z$ are connected due to the assumed linearity of Z. Bearing in mind that the sections f^y, $y \in Y$ are nonalternating, we conclude that $(f^y)^{-1}(\bar{B}(g(y),s))$ is connected and thus also convex. Hence $h^{-1}(\bar{B}(g,s))$ is convex being the intersection of the indexed family of convex sets. Since each convex set on $\mathbb{R}$ is ambiguous, $h^{-1}(U) \in F_\sigma(\mathbb{R})$ for each open subset $U \subset h(\mathbb{R})$ provided U is a countable union of closed balls. Consequently $h : X \to B_1(Y,Z)$ is of the first class of Baire and has separable range. Observe that $f(x,y) = h(x)(y)$ so that, by Baire's Theorem, the Y-sections of f satisfy property A_2. Invoking Theorem 0 with $\alpha = 1$ we obtain the claimed assertion.

REMARK 3. Note that the space $\mathbb{R}$ may be generalized to be, for example, a curve in Euclidean space, in particular a circle, i.e. a topological space having no order compatible with the topology.

COROLLARY 3. Assume additionally that Y is a compact metric space. Let $f : \mathbb{R} \times Y \to Z$ be a transformation with nonalternating Y-sections and continuous X-sections. Then f is in the first Baire class.

PROOF: The space $C(Y,Z)$ endowed with the metric $D(g_1,g_2) :=$ $:= \sup\{d(g_1(y),g_2(y)) : y \in Y\}$ is separable due to the compactness of Y and the separability of Z. Thus we may apply Proposition 2.

REMARK 4. The space Y in Corollary 3 may be generalized to be a "chunk-complex" (See [2], p. 118.), i.e. a topological space having a family $\{K_a : a \in A\}$ of closed subsets, such that:

(i) $\{K_a : a \in A\}$ is a covering of Y,

(ii) either $K_a \cap K_b = \emptyset$ or $K_a \cap K_b = K_c$ for some $c \in A$,

(iii) $\{a \in A : K_a \subset K_b\}$ is finite for each $b \in A$,

(iv) each K_a is compactly metrizable by some metric d_a,

(v) U is open in Y iff $K_a \cap U$ is open in (K_a,d_a) for all $a \in A$.

In fact, $f|_{\mathbb{R}\times K_a}$ is Baire 1 by virtue of Proposition 2. The space Y being paracompact and perfectly normal (cf. [2]), we may apply Baire's theorem (cf. [8]) and the property (v) to conclude that f is Baire 1 on the entire space $\mathbb{R} \times Y$. Note that in particular $\mathbb{R}$ is chunk-complex and thus in case $Z = \mathbb{R}$ Corollary 3 gives a negative answer to question 3 a), g) from [5].

In connection with Corollary 3 let us recall that by an old result of H.D. Ursell cited in [4], a function $f : \mathbb{R}^2 \to \mathbb{R}$ with isotonic Y-sections and L measurable X-sections is L measurable on the plane. Obviously this result may be improved in the style of Proposition 2. On the other hand, a function $f : \mathbb{R}^2 \to \mathbb{R}$ with all X-sections and all Y-sections nondecreasing may fail to be Borel measurable. In fact, let us decompose $\mathbb{R}$ into two disjoint nonmeasurable subsets A and B and then put:

$$f(x,y) := \begin{cases} 0 & \text{if } y < -x \\ 3 & \text{if } y > -x \\ 2 & \text{if } x = -y \in A \\ 1 & \text{if } x = -y \in B \end{cases} .$$

It is easily seen that f is as required. Next it is well known that a separately continuous, real function whose Y-sections are in addition isotonic is jointly continuous. (See for example [3].) It would be interesting to known whether or not Corollary 3 remains true with the condition on the Y-sections weakened to bounded variation and without the compactness assumption imposed on Y.

Our last proposition gives a positive answer to questions 3 b), d) from [5] and shows the sharpness of the known result, that a function $f : \mathbb{R}^2 \to \mathbb{R}$ with right or left continuous X-sections and Baire 1 Y-sections belongs to Baire class 2:

PROPOSITION 3. Let $I := [0,1]$. There exists a function $f : I^2 \to \mathbb{R}$ not belonging to the first Baire class for which all X-sections are left continuous and increasing while all Y-sections are decreasing.

Proof: Let C be the Cantor ternary set in I and $\{(a_n,b_n) : n \in \mathbb{N}\}$ its contiguous intervals. Let us consider the triangles $T_0 := \operatorname{conv}\{(0,0),(1,0),(1,1)\}$ and $T_n := \operatorname{conv}\{(a_n,a_n),(b_n,a_n),(b_n,b_n)\} \subset I^2$, $n \in \mathbb{N}$. Then put:

$$f(x,y) := \begin{cases} \dfrac{y-a_n}{b_n-a_n} & \text{if } (x,y) \in T_n\,, \quad n \in \mathbb{N}\,, \\ 0 & \text{if } (x,y) \in T_0 \setminus \bigcup_{n=1}^{\infty} T_n \\ +1 & \text{if } (x,y) \in I^2 \setminus T_0 \end{cases}$$

Observe that $\lim_{t \to y_-} f(x,t) = f(x,y)$ and that $u \geq v$ implies $f(x,v) \leq f(x,u)$ and $f(u,y) \leq f(v,y)$ whenever $(x,y) \in I^2$. Define a perfect set $P := \{(t,t) \in I^2 : t \in C\}$ and observe that the fibers $(f|p)^{-1}(\{1\}) = \{(b_n,b_n) : n \in \mathbb{N}\}$ and $(f|_p)^{-1}(\{0\}) = \{(a_n,a_n) : n \in \mathbb{N}\}$ are both dense in P. Thus the restriction $f|p$ is totally discontinuous so that by Baire's Theorem, f cannot be in the first class.

The following question 3 c), e) from [5] remains unresolved: Let all X-sections of $f : \mathbb{R}^2 \to \mathbb{R}$ be derivatives (resp. bounded derivatives, approximately continuous, etc.) and all Y-sections be increasing. Must then f belong to the first Baire class?

In connection with this problem let us mention that there is a function with continuous X-sections and Darboux Baire 1 Y-sections which is not in the first class [5]. On the other hand the continuity of X-sections and approximate continuity of Y-sections implies that f is of the first class. (See (iii) in [7].) However this is a solution of question 2 a) from [5]. If all X-sections are bounded derivatives and all Y-sections are in the first class, then f must be in the second class, but there exists a function with continuous Y-sections all of whose X-sections are derivatives and yet not belonging to the first class of Baire [7]. This solves in the affirmative question 2 d), f) from [5].

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