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FUNCTIONS THAT NEARLY PRESERVE G_δ-SETS

1. <u>Introduction</u>. By a G_δ-subset of the real line R we mean the intersection of a countable family of open sets in R. In real analysis it is of interest to note that certain kinds of functions map G_δ-sets to G_δ-sets. For example, the range of a homeomorphic mapping from a G_δ-set X to the real line must be a G_δ-set [1, Theorem 63]. More generally, the range of a one-to-one continuous mapping from a Borel set X to the real line must be a Borel set [1, Theorem 87]. We pose the question: what kind of functions mapping the unit interval [0,1] into R map each G_δ-set to a G_δ-set? This turns out not to be a good question, because most classes of functions we study are likely to contain members that map some G_δ-set to a set that is not a G_δ-set. So we make a slight modification.

Definition. Let f be a real valued function defined on [0,1]. We say that f is a <u>delta</u> function if for each G_δ-set $X \subset [0,1]$, f(X) is the union of a G_δ-set with a countable set.

We will study which continuous functions on [0,1] are delta functions and which functions of bounded variation on [0,1] are delta functions. We will find that a necessary and sufficient condition for a continuous function f on [0,1] to be a delta function is that for all but countably many y, $f^{-1}(y)$ is a finite set (Theorem 1). This does not work when "function of bounded variation" replaces "continuous function". But if f is of bounded variation and if $f^{-1}(y)$ is a singleton set for all but countably many $y \in f[0,1]$, then f is a delta function (Theorem 4). We conclude with some examples of delta functions that do not map every G_δ-set to a G_δ-set.

__2__. __Continuous functions__. We begin with some nuts and bolts lemmas on subsets of R and continuous functions on $[0,1]$. The first is closely related to standard arguments.

Lemma 1. Let X be a perfect subset of R and let I be an open interval that meets X. Then there exists an uncountable closed subset of $I \cap X$ that is a first category set relative to the subspace X.

The proof is long but straight-forward, so we leave it. The next lemma is even easier.

Lemma 2. Let E be an uncountable subset of an interval J. Then there exist disjoint compact subintervals J_1 and J_2 of J such that $E \cap J_1$ and $E \cap J_2$ are uncountable.

Proof. Let U denote the union of all open intervals I with rational endpoints for which $E \cap I$ is countable. Then $E \cap U$ is countable and $J \setminus U$ is uncountable. Now let J_1 and J_2 be disjoint compact subintervals of J, each centered at a point in $J \setminus U$. Then $E \cap J_1$ and $E \cap J_2$ must be uncountable. □

We turn now to G_δ-sets.

Lemma 3. Let (U_n) be a sequence of mutually disjoint open sets and for each n let E_n be a G_δ-set with $E_n \subset U_n$. Then $\bigcup_n E_n$ is also a G_δ-set.

Proof. For each n, let $(V_{nj})_j$ be a sequence of open sets such that $V_{nj} \subset U_n$ and $\bigcap_j V_{nj} = E_n$. Then for each j,

$$\bigcup_n V_{nj} \subset \bigcup_n E_n \text{ and moreover } \bigcup_n [\bigcap_j V_{nj}] = \bigcup_n E_n .$$

Because the set U_n are mutually disjoint, $V_{nj} \cap V_{n'j'} = \emptyset$ for $n \neq n'$, and it follows that

$$\bigcup_n E_n = \bigcup_n [\bigcap_n V_{nj}] = \bigcup_j [\bigcap_n V_{nj}] .$$

Because each $\bigcup_n V_{nj}$ is open, $\bigcup_n E_n$ is a G_δ-set. □

Until further notice we assume that f is a continuous function on $[0,1]$ such that there are uncountably many points y for which $f^{-1}(y)$ is an infinite set.

Lemma 4. Let U be an open subset of $[0,1]$ and let I be a compact interval with $I \subset [0,1] \setminus U$. Let there be uncountably many $y \in f(I)$ such that $U \cap f^{-1}(y)$ is an infinite set. Then there exist disjoint compact intervals I_1 and I_2 with $I_1 \cup I_2 \subset U$ such that $f(I_1) \cup f(I_2) \subset f(I)$, $f(I_1) \cap f(I_2) = \emptyset$, and for each $k = 1,2$, there are uncountably many $y \in f(I_k)$ for which $(U\backslash(I_1 \cup I_2)) \cap f^{-1}(y)$ is an infinite set, and length $f(I_k) < \frac{1}{2}$ length $f(I)$.

Proof. By Lemma 2, there are disjoint compact subintervals J_1 and J_2 of the interior of $f(I)$ such that for each $k = 1,2$, there are uncountably many $y \in J_k$ for which $U \cap f^{-1}(y)$ is an infinite set. Because f is uniformly continuous on $[0,1]$ there is a number $c > 0$ such that for no interval K of length $< c$ can an interval twice the length of $f(K)$ meet both J_1 and J_2, or meet both $J_1 \cup J_2$ and the complement of $f(I)$; hence length $f(K) < \frac{1}{2}$ length $f(I)$ as well.

For each positive integer n, partition U into a countable collection of mutually disjoint intervals $K_{n1}, K_{n2}, K_{n3}, \ldots$, each of length $< c/n$. Let E denote the set of all $y \in J_1$ for which $U \cap f^{-1}(y)$ is an infinite set. For each $y \in E$ there is an index n for which $f^{-1}(y)$ meets 2 intervals K_{ni}. So there is an index N such that for uncountably many $y \in E$, $f^{-1}(y)$ meets 2 intervals K_{Ni}. Thus there exist indices i and i' such that for uncountably many $y \in E$, $f^{-1}(y)$ meets K_{Ni} and $K_{Ni'}$. By making $I_1 = K_{Ni}$ or $K_{Ni'}$, whichever is appropriate, we have an interval $I_1 \subset U$ of length $< c$, such that there are uncountably many $y \in E \cap f(I_1)$ for which $(U\backslash I_1) \cap f^{-1}(y)$ is an infinite set and moreover $f(I_1) \cap J_1 \neq \emptyset$. From the choice of c it follows that $f(I_1) \subset f(I)$, and length $f(I_1) < \frac{1}{2}$ length $f(I)$.

Similarly there is an interval $I_2 \subset U$ of length $< c$, such that there are uncountably many $y \in E \cap f(I_2)$ for which $(U\backslash I_2) \cap f^{-1}(y)$ is an infinite set and moreover $f(I_2) \cap J_2 \neq \emptyset$. It follows from the choice of c that $f(I_2) \subset f(I)$, $f(I_1) \cap f(I_2) = \emptyset$, and length $f(I_2) < \frac{1}{2}$ length $f(I)$. Hence I_1 and I_2 are the desired intervals. □

We next construct a perfect set in the range of f.

Lemma 5. Let f be a continuous function on $[0,1]$ and let there be uncountably many $y \in f[0,1]$ such that $f^{-1}(y)$ is an infinite set. Then there exists a family of mutually disjoint compact subintervals $\{I_a\}$ of $[0,1]$ on which f is not constant, where each subscript a is a finite sequence of 1s and 2s; moreover $f(I_a) \subset f(I_b)$ and length $f(I_a) < \frac{1}{2}$ length $f(I_b)$ if

b is an initial segment of a, and $f(I_a) \cap f(I_b) = \emptyset$ if neither a nor b is an initial segment of the other.

Proof. Let E be the set of all $y \in f[0,1]$ such that $f^{-1}(y)$ is an infinite set that contains no interval. Then E is uncountable. For each positive integer n, partitition $[0,1]$ into a countable collection of mutually disjoint intervals $K_{n1},K_{n2},K_{n3},\ldots$, each of length $< 1/n$. For each $y \in E$ there is an index n for which $f^{-1}(y)$ meets 2 intervals K_{ni}. So there is an index N such that for uncountably many $y \in E$, $f^{-1}(y)$ meets 2 intervals K_{Ni}. Thus there exist indices i and i' such that for uncountably many $y \in E$, $f^{-1}(y)$ meets K_{Ni} and $K_{Ni'}$. By making $I = K_{Ni}$ or $K_{Ni'}$, whichever is appropriate, we have an interval I such that there are uncountably many $y \in E \cap f(I)$ for which $((0,1)\backslash I) \cap f^{-1}(y)$ is an infinite set.

By Lemma 4, there exist disjoint compact intervals I_1 and I_2 with $I_1 \cup I_2 \subset (0,1)\backslash I$ such that $f(I_1) \cup f(I_2) \subset f(I)$, $f(I_1) \cap f(I_2) = \emptyset$, and for each $k = 1,2$, there are uncountably many $y \in f(I_k)$ for which $((0,1)\backslash(I \cup I_1 \cup I_2)) \cap f^{-1}(y)$ is an infinite set, and length $f(I_k) < \frac{1}{2}$ length $f(I)$.

By Lemma 4, for each $j = 1,2$, there exist disjoint compact intervals I_{j1} and I_{j2} with $I_{j1} \cup I_{j2} \subset (0,1)\backslash(I \cup I_1 \cup I_2)$ such that $f(I_{j1} \cup I_{j2}) \subset f(I_j)$, $f(I_{j1}) \cap f(I_{j2}) = \emptyset$, and for each $k = 1,2$, there are uncountably may $y \in E \cap f(I_{jk})$ for which $((0,1)\backslash(I \cup I_1 \cup I_2 \cup I_{11} \cup I_{12} \cup I_{21} \cup I_{22})) \cap f^{-1}(y)$ is an infinite set, and length $f(I_{jk}) < \frac{1}{2}$ length $f(I_j)$.

By Lemma 4, for each $j = 1,2$ and $k = 1,2$, there exist disjoint compact intervals I_{jk1} and I_{jk2} with $I_{jk1} \cup I_{jk2} \subset (0,1)\backslash(I \cup I_1 \cup I_2 \cup I_{11} \cup I_{12} \cup I_{21} \cup I_{22})$ such that $f(I_{jk1} \cup I_{jk2}) \subset f(I_{jk})$, $f(I_{jk1}) \cap f(I_{jk2}) = \emptyset$, and for each $i = 1,2$, there are uncountably many $y \in E \cap f(I_{jki})$ for which $((0,1)\backslash(I \cup I_1 \cup I_2 \cup I_{11} \cup I_{12} \cup I_{21} \cup I_{22} \cup (\bigcup_{i,j,k=1}^{2} I_{jki})) \cap f^{-1}(y)$ is an infinite set, and length $f(I_{jki}) < \frac{1}{2}$ length $f(I_{jk})$.

We continue to use Lemma 4 in this way, so for each finite sequence a, we select disjoint compact intervals I_{a1} and I_{a2} disjoint from all the intervals previously selected such that $f(I_{a1} \cup I_{a2}) \subset f(I_a)$, $f(I_{a1}) \cap f(I_{a2}) = \emptyset$, and for each $i = 1,2$, length $f(I_{ai}) < \frac{1}{2}$ length $f(I_a)$ and there are uncountably many $y \in E \cap f(I_{ai})$ for which infinitely many points of $f^{-1}(y)$ lie outside all the previously selected intervals and outside $I_{a1} \cup I_{a2}$.

□

Lemma 6. Let f be a continuous function on $[0,1]$ and let there be uncountably many $y \in f[0,1]$ such that $f^{-1}(y)$ is an infinite set. Then there is a G_δ-set $X \subset [0,1]$ such that $f(X)$ is not the union of a G_δ-set with a countable set.

Proof. Let the intervals I_a be as in Lemma 5, and let $I_a = [r_a,s_a]$. Let Y consist of all points y that lie in $f(I_a)$ for infinitely many indices a. For fixed n, $Y \subset \bigcup_a f(I_a)$ where a runs over those sequences of length n. It follows that Y is the intersection of a contracting sequence of nonvoid compact sets, and Y is likewise nonvoid and compact. Length $f(I_a)$ tends to 0 as the length of the sequence a tends to ∞. It follows from the construction that Y has no isolated points, and Y is a perfect set.

For each sequence a, use Lemma 1 to construct a nonvoid closed uncountable subset Y_a of $Y \cap f(I_a)$ that is a first category subset of Y. Then $[r_a,s_a] \cap f^{-1}(Y_a)$ is a closed set and $(r_a,s_a) \cap f^{-1}(Y_a)$ is a G_δ-set. Put $X = \bigcup_a (r_a,s_a) \cap f^{-1}(Y_a)$. By Lemma 3, X is a G_δ-set. Moreover $f(X)$ is a first category set relative to the subspace Y, and every point of Y is a condensation point of $f(X)$. Consequently for any countable set C, $f(X)\backslash C$ is a dense first category set relative to Y. Thus $f(X)\backslash C$ cannot be a G_δ-set relative to Y; but Y is a subspace of R, so $f(X)\backslash C$ is not a G_δ-set relative to R. Finally, X is a G_δ-set, but $f(X)$ is not the union of a G_δ-set with a countable set in R. □

Lemma 6 proves half of our first result. The converse will be easy.

Theorem 1. Let f be a continuous function on $[0,1]$. Then f is a delta function if and only if for all but at most countably many points y, $f^{-1}(y)$ is a finite set.

Proof. If there exist uncountably many y for which $f^{-1}(y)$ is an infinite set, then by Lemma 6, f is not a delta function.

Now assume there are only countably many y for which $f^{-1}(y)$ is an infinite set. Let $U \subset [0,1]$ be an open set. Then f maps each component interval of U to a connected set; i.e., to the union of an open set and a finite set. It follows that $f(U)$ is the union of an open set and a countable set.

Now let X be a G_δ-subset of $[0,1]$. Let $U_1 \supset U_2 \supset U_3 \supset \cdots$ be a contracting sequence of open sets such that $X = \bigcap_n U_n$. Let $f(U_n) = V_n \cup C_n$ where V_n is open and C_n is a countable set. Then

$$f(\bigcap_n U_n) \subset \bigcap_n f(U_n)$$

and any point y in the difference of these sets must have infinitely many points in $f^{-1}(y)$. Thus

$$f(\bigcap_n U_n) \cup (\text{countable set}) = \bigcap_n f(U_n) \quad .$$

But $\bigcap_n f(U_n) = \bigcap_n(V_n \cup C_n) = [\bigcap_n V_n] \cup (\text{countable set})$. It follows that

$$f(X) = f(\bigcap_n U_n) = \{[\bigcap_n V_n] \setminus (\text{countable set})\} \cup (\text{countable set}) \quad .$$

But $\bigcap_n V_n \setminus (\text{countable set})$ is a G_δ-set because each V_n is open. So f is a delta function. □

By a nowhere monotonic function on $[0,1]$ we mean a function on $[0,1]$ that is not monotonic on any subinterval of $[0,1]$. Routine arguments (we will leave) show that if f is continuous, nowhere monotonic, the points y for which $f^{-1}(y)$ is an infinite set, is a second category set in R. Thus f cannot be a delta function. Indeed if f is nowhere monotonic on any subinterval of $[0,1]$, f cannot be a delta function.

So if f is a continuous delta function on $[0,1]$ then any subinterval of $[0,1]$ meets an interval on which f is monotonic; thus f is differentiable on a dense subset of $[0,1]$. Before we give another application of delta functions we need a lemma on intervals in R.

Lemma 7. Let $c > 0$. Then there exists a sequence $(I_n)_{n=0}^{\infty}$ of mutually disjoint closed subintervals of $[0,1]$ such that the left endpoint of I_0 is 0, the right endpoint of I_1 is 1, Σ_n (length I_n) $< c$, the set $[0,1]\setminus\bigcup_n I_n$ is dense in itself, the set $\bigcup_n I_n$ is dense in $[0,1]$, and for each $n \geq 2$, the midpoint of I_n is also the midpoint of the interval joining the midpoints of the two intervals among $I_0, I_1, \ldots, I_{n-1}$ that I_n lies between.

The proof is a straight-forward inductive construction, so we leave it.

Theorem 2. Let $c > 0$. Then there is a continuous delta function f on $[0,1]$ such that the measure of the set of all points where f is differentiable is $< c$.

Proof. Let $(I_n)_{n=0}^{\infty}$ be the intervals in Lemma 7. We define functions g_n for each $n \geq 2$ as follows. If $n \geq 2$ and $I_n = [a,b]$, make $g_n(\frac{1}{2}(a + b))$ = the distance between the midpoints of the two intervals among $I_0, I_1, \ldots, I_{n-1}$ that I_n lies between. Make $g_n[0,a] = 0 = g_n[b,1]$, and make g_n linear on $[a,\frac{1}{2}(a + b)]$ and $[\frac{1}{2}(a + b),b]$. Thus each g_n is a continuous function on $[0,1]$ and $f = \Sigma_{n=2}^{\infty} g_n$ is also continuous on $[0,1]$. Moreover $f^{-1}(y)$ is a finite set for $y \neq 0$. By Theorem 1, f is a delta function.

It remains only to prove that f is not differentiable at any $x \in (0,1)\setminus\cup_{n=0}^{\infty} I_n$. There are infinitely many indices n such that no interval among $I_0, I_1, \ldots, I_{n-1}$ lies between x and I_n. We obtain from construction, $|f(x) - f(p)| > |x - p|$ where p is the midpoint of I_n. On the other hand, if $q \in (0,1)\setminus\cup_{n=0}^{\infty} I_n$, then $f(x) - f(q) = 0$. Because x is an accumulation point of $(0,1)\setminus\cup_{n=0}^{\infty} I_n$, f is not differentiable at x. □

3. Bounded variation. In this section f will be a function of bounded variation on the interval $[0,1]$. Such a function can be discontinuous at only countably many points, but these points may make a considerable difference. For example, Theorem 1 is not in general true when "continuous function" is replaced by "function of bounded variation." Indeed there exist functions of bounded variation f that are not delta functions, such that $f^{-1}(y)$ has more than two points for no y.

Theorem 3 There exists a function f on bounded variation on $[0,1]$ such that f is not a delta function and for each point y, $f^{-1}(y)$ is at most a doubleton set. Moreover, there is an open subset U of $[0,1]$ such that $f(U)$ is not the union of a G_δ-set with a countable set.

Proof. Let C denote the Cantor set. Each point of C is uniquely expressed as the sum $\Sigma_{n=1}^{\infty} (2a_n)3^{-n}$, where (a_n) is a sequence of 0s and 1s. Let I be a complementary interval of C of the form $(x, x + 3^{-2k-1})$ where $x = \Sigma_{n=1}^{2k} (2a_n)3^{-n} + 3^{-2k-1} \in C$. Let U be the union of all such intervals. Then U is an open dense subset of $[0,1]$ disjoint from C.

Let I be as in the preceding paragraph and let $u \in I$. Let $u - x = \Sigma_{n=1}^{\infty} b_n 2^{-n}$ where each $b_n = 0$ or 1 and $b_n = 1$ for infinitely many n. Define $g(u) = x + \Sigma_{n=1}^{\infty} (2b_n)3^{-4k-2n} \in I$ and $f(u) = g(u) + 3^{-2k-1} \in C$. For $t \in [0,1]\setminus U$ define $g(t) = f(t) = t$. Then g is increasing on $[0,1]$ and has total variation 1. Moreover, the total variation of f is $\leq 1 + 2\,\Sigma_I\, 3^{-2k-1} = 1 + 2(\text{measure of } U) < 3$, and f has bounded variation on $[0,1]$. If $u_1, u_2 \in U$ and $f(u_1) = f(u_2)$, note that the component interval of U containing u_i is determined by the last nonzero term of the expansion of $f(u_i)$ in which the power of 3 is odd, and the terms that precede it; thus u_1 and u_2 lie in the same component interval and it follows that $g(u_1) = g(u_2)$, $u_1 - x = u_2 - x$ and $u_1 = u_2$. Consequently $f^{-1}(y)$ is at most a doubleton set for each point y. Clearly $f(x, x + 3^{-2k-1})$ is an uncountable nowhere dense subset of C, and $f(U)$ is a first category set relative to C such that each point of C is a condensation point of $f(U)$.

If D is any countable set, $f(U)\setminus D$ is a dense first category set of C and is not a G_δ-set relative to C, or to R. Thus $f(U)$ is not the union of a countable set with a G_δ-set relative to R, and f is not a delta function. □

Before we give a sufficient condition for a function of bounded variation to be a delta function, we need more notation. Let f be a function on the interval $[0,1]$. We say that f is an L-function if $f(x+)$ exists for each $x \in [0,1)$ and if $f(x-)$ exists for each $x \in (0,1]$. (We admit infinite limits.)

We note that if f is an L-function, then f has at most countably many points of discontinuity. For take any $\varepsilon > 0$, and set $g = \arctan f$. It is easy to see that the set $\{x \in (0,1) : |g(x+) - g(x)| + |g(x) - g(x-)| > \varepsilon\}$ is finite. It follows that the set of all points of discontinuity of g and hence of f is countable.

Lemma 8. Let f be an L-function and let X be a subinterval of $[0,1]$. Then $\overline{f(X)}\setminus f(X)$ is countable.

Proof. Let $a = \inf X$, $b = \sup X$. Let D be the set of all points of discontinuity of f in (a,b); let $T_+ = \{f(x+) : x \in D\}$, $T_- = \{f(x-) : x \in D\}$, $V = \{f(a+), f(b-)\}$. Let $y \in \overline{f(X)}\setminus f(X)$. There are $x_n \in X$ such that $f(x_n) \to y$. Let (t_n) be a monotone subsequence of (x_n) and let $t_n \to t$. Because $\lim f(t_n) = y \notin f(X)$, we have $t_n \neq t$ for each n. If $t \in (a,b)$, then $t \in D$ and $y \in T_+ \cup T_-$; if $t \in \{a,b\}$, then $y \in V$. Hence $\overline{f(X)}\setminus f(x) \subset T_+ \cup T_- \cup V$ which is countable. □

We offer more notation.

(1) For any sets P and Q, let $P \sim Q$ mean that $(P\setminus Q) \cup (Q\setminus P)$ is countable.

(2) Let $\mathfrak{a}$ denote the system of all subsets of R that can be expressed as the union of a G_δ-set and a countable set.

Lemma 9. (1) If $P \sim Q$ and $Q \in \mathcal{A}$, then $P \in \mathcal{A}$.

(2) If $P_1, P_2, P_3, \ldots \in \mathcal{A}$, then $\cap P_n \in \mathcal{A}$.

We leave the proof.

Theorem 4. Let f be an L-function and let the set

$$S = \{y \in R : f^{-1}(y) \text{ has at least two elements}\}$$

be countable. Then f is a delta function. Moreover, if f is strictly monotonic on $[0,1]$, then f maps G_δ-sets in $[0,1]$ to G_δ-sets in R.

Proof. Let A be an open set in $[0,1]$ and let $B = [0,1]\backslash A$. There are intervals $J_1, J_2, J_3, \ldots$ such that $A = \cup J_n$. Set $F = \overline{f[0,1]}$, $K_n = [0,1]\backslash J_n$, $F_n = \overline{f(K_n)}$. According to Lemma 8 we have $f[0,1] \sim F$ and $f(K_n) \sim F_n$ $(n = 1,2,3,\ldots)$. It follows that $\cap f(K_n) \sim \cap F_n$. Since S is countable, we have $F(A) \sim f[0,1]\backslash f(B)$ and $f(B) = f(\cap K_n) \sim \cap f(I_n)$. Hence $f(A) \sim F\backslash \cap F_n$ which is a G_δ-set. Thus $f(A) \in \mathcal{A}$.

Now let $A = \cap A_n$, where $A_1, A_2, A_3, \ldots$ are open in $[0,1]$. Since S is countable, we have $f(A) \sim \cap f(A_n)$. Applying what has just been proved and Lemma 9 we get $f(A) \in \mathcal{A}$ which completes the proof that f is a delta function.

Finally, let f be strictly monotonic on $[0,1]$. For any interval J, $f(J)$ and $\overline{f(J)}$ differ by a countable set by Lemma 8, so $f(J)$ is a G_δ-set. In the notation of the preceding two paragraphs, $f(A) = \cup f(J_n)$ is a G_δ-set by Lemma 3, and $f(A) = \cap f(A_n)$ is a G_δ-set because f is one-to-one. □

Unfortunately the condition in Theorem 4 is not necessary for f to be a delta function. Witness the delta function $f(x) = |x - \frac{1}{2}|$.

Does there exist a one-to-one function of bounded variation on $[0,1]$ that does not map every G_δ-set to a G_δ-set? This is a natural question in view of Theorem 4. The following example shows that the answer is yes.

Let C denote the Cantor set and let U denote the open set $(0,1)\backslash C$. Let $U_1, U_2, U_3, \ldots$ denote the components of U, let the left endpoint of U_n be a_n and the midpoint of U_n be d_n. For each n, define $f(a_n) = d_n$ and $f(d_n) = a_n$. For all other points t, define $f(t) = t$. Then f is a one-to-one function on $I = [0,1]$. It is easy to show (and we leave it) that the total variation of f is $\leq 1 + 4(\text{measure of } U) = 5$, so f is of bounded variation on I. Now $C \cap f(U) = \{a_n\}$ is a countable dense subset of C and is not a G_δ-set relative to C or to R. So $f(U)$ is not a G_δ-set.

Does there exist a continuous function on I that is one-to-one and does not map every G_δ-set to a G_δ-set? No, because a one-to-one continuous function on I must be strictly monotonic. It is not difficult to show that if f is continuous on I and if for each point y, $f^{-1}(y)$ contains at most two points, then f is piecewise strictly monotonic. However we do have the following example. Let ϕ be an infinitely many times differentiable function on R such that $\phi = 0$ on $(-\infty,0] \cup [1,\infty)$, $\phi > 0$ on $(0,1/3) \cup (2/3,1)$, $\phi < 0$ on $(1/3,2/3)$, $\int_0^{1/3} \phi < \int_{2/3}^1 \phi$ and $\int_0^{2/3} \phi = 0$. Let C be the Cantor set and let $U_n = (a_n,b_n)$ be disjoint intervals such that $\cup U_n = I\backslash C$ and let $\varepsilon_n = b_n - a_n$ $(n = 1,2,3,\dots)$. Set $\psi(x) = \Sigma_{n=1}^{\infty} n^{-2}\varepsilon_n^n \phi((x - a_n)/\varepsilon_n)$ and $f(x) = \int_0^x \psi(x \in I)$. It can be shown that f is infinitely many times differentiable, $f^{-1}(y)$ contains at most three points for each y and f is a delta function by Theorem 1. As in the preceding paragraph, $f(I\backslash C)$ is not a G_δ-set in R.

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