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LATTICES, ALGEBRAS AND BAIRE'S SYSTEMS GENERATED BY SOME FAMILIES OF FUNCTIONS

I. Preliminaries. Let us establish some of the terminology to be used. R denotes the real line. Let (X,T) be a topological space. A function $f:X \to R$ is said to be T-quasi-continuous at a point $x_0 \in X$ iff for every $\varepsilon > 0$ and for any neighbourhood $U \in T$ of the point x_0 there exists a T-open set V such that $0 \neq V \subset U$ and $|f(x) - f(x_0)| < \varepsilon$ for every $x \in V$, T-cliquish at $x_0 \in X$ iff for every $\varepsilon > 0$ and for any neighbourhood $U \in T$ of the point x_0 there exists a T-open set V such that $0 \neq V \subset U$ and $|f(x) - f(x_1)| < \varepsilon$ for $x, x_1 \in V$.

A function $f:X \to R$ is T-quasi-continuous (T-cliquish) on X iff f is T-quasi-continuous (T-cliquish) at every point of X.

Let $X = R^m$. We shall use the following differentiation basis. For every $k \in N$ (N denotes the set of all positive integers) let P_k be the family of all m-dimensional intervals of the form

$$\langle i_1-1/2^k, i_1/2^k) \times \dots \times \langle i_m-1/2^k, i_m/2^k)$$

where $i_1, i_2, \dots, i_m = 0, \pm 1, \pm 2, \dots$.

Let $\mathcal{P} = \bigcup_{k=1}^{\infty} P_k$ .Let $A \subset R^m$ be a set. For $x \in R^m$ we can define the upper outer density of A at a point x by

$$\bar{d}(A,x) = \overline{\lim_{\substack{P \Rightarrow x \\ P \in \mathcal{P}}}} |A \cap P|/|P| ,$$

where $|A|$ denotes m-dimensional Lebesgue outer measure of A and the understanding of the symbol $P \Rightarrow x$ is that $x \in P$ and the diameter of P tends to zero.

Denote by T_e the Euclidean topology in R^m and by d_2 the density topology relative to the differentiation basis $(\mathcal{P}, \Rightarrow)$.

The symbols Q_T, Cq_T stand for the family of all T-quasi-continuous functions $f: R^m \to R$ and the family of all T-cliquish functions, respectively.Evidently we have $Q_T \subset Cq_T$.

If K is a family of functions $f: X \to R$ then

(i) A(K) denotes the algebra generated by K, i.e. the least family for which: $K \subset A(K)$, $f+g \in A(K)$, $f.g \in A(K)$ for any $f,g \in A(K)$;

(ii) B(K) denotes the collection of all pointwise limits of sequences taken from K ;

(iii) L(K) denotes the lattice generated by K, i.e. the least family for which $\max(f,g) \in L(K)$ and $\min(f,g) \in L(K)$ for any $f,g \in L(K)$.

Let $(w_n)_n$ be an ennumeration of all rationals with $w_0 = 0$.

II. Results.

Theorem 1. A function $f: R^m \to R$ is d_2-cliquish iff f is Lebesgue measurable.

Definition 1. A measurable function $f: R^m \to R$ is degenerate at a point $x_0 \in R^m$ iff there exists a neighbourhood U of $f(x_0)$ such that the set $f^{-1}(U)$ has the density zero at x_0.
A measurable function f is nondegenerate iff it is not degenerate at any point.

Theorem 2. A Lebesgue measurable function $f: R^m \to R$ is d_2-quasi-continuous iff f is nondegenerate.

Basic lemma. Assume that $A \subset R^m$ is a G_δ set of Lebesgue measure zero, $G \subset R^m$ is an open set and $A \subset G$. Then there exists a sequence of pairwise disjoint (L) measurable sets $A_n \subset G - A$ $(n = 0,1,2,\dots)$ such that $\bigcup_{n=0}^{\infty} A_n = G - A$, $\overline{d}(A_n, x) > 0$ for every $x \in A \cup A_n$ $(n = 0,1,2,\dots)$ and $\overline{d}((R^m - G) \cup A_0, x) > 0$ for each $x \in R^m - G$.

Theorem 3. $A(Q_{d_2}) = Cq_{d_2}$.

An outline of proof. It is enought to prove that $Cq_{d_2} \subset A(Q_{d_2})$.

Let $f \in Cq_{d_2}$. Let A be a G_δ set of measure zero which contains the set of all d_2-discontinuity points of f. Let A_n $(n=0,1,2,\dots)$ be sets satisfy the conclusion of Basic lemma (for $G = R^m$).
Let us put

$$f_1(x) = \begin{cases} f(x) & \text{for } x \in A \\ w_n & \text{for } x \in A_{2n} \quad ; \ n=0,1,2,\dots \\ f(x) - w_n & \text{for } x \in A_{2n+1} \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{for } x \in A \\ f(x) - w_n & \text{for } x \in A_{2n} \ ; \ n=0,1,2,\ldots \\ w_n & \text{for } x \in A_{2n+1} \end{cases}$$

The functions f_1, f_2 are d_2-quasi-continuous and $f = f_1 + f_2$.

Theorem 4. $L(Q_{d_2}) = Cq_{d_2}$

An outline of proof. For $f \in Cq_{d_2}$ and $i=0,1,2,3$ let us put

$$f_i(x) = \begin{cases} w_n & \text{for } x \in \bigcup_{n=0}^{\infty} A_{4n+i} \\ f(x) & \text{for } x \notin \bigcup_{n=0}^{\infty} A_{4n+i} \end{cases}$$

The functions f_i $(i = 0,1,2,3)$ are d_2-quasi-continuous and

$$f = \min\left(\max\left(f_0, f_1\right), \max\left(f_2, f_3\right)\right)$$

Theorem 5. $B(Q_{d_2}) = Cq_{d_2}$

An outline of proof. It is enought to prove that $Cq_{d_2} \subset B(Q_{d_2})$. If $f \in Cq_{d_2}$ then there exists a Baire 2 function $g: R^m \longrightarrow R$ a.e. equal to f. Let $h = f - g$ and let $(G_n)_n$ be a decreasing sequence of open sets such that $\bigcap_{n=1}^{\infty} G_n = A \supset \{x \in R^m;\ h(x) \neq 0\}$ et $|A| = 0$. For $n=1,2,\ldots$ let $(A_{nk})_k$ be a sequence of measurable sets which satisfies the conclusion of Basic lemma (for $G = G_n$). Define

$$h_n(x) = \begin{cases} w_k & \text{for } x \in \bigcup_{n=1}^{\infty} A_{nk} \\ h(x) & \text{for } x \in A \\ 0 & \text{for } x \in \left(R^m - G_n\right) \cup A_{n0} \end{cases}$$

The functions h_n $(n=1,2,\ldots)$ are d_2-quasi-continuous and $h = \lim_{n\to\infty} h_n$. Since g is Baire 2, there exists a sequence (g_n) of d_2-continuous functions with $g = \lim_{n\to\infty} g_n$. The sum $h_n + g_n$ $(n = 1,2,\ldots)$ is d_2-quasi-continuous and $f = g + h = \lim_{n\to\infty}(g_n + h_n)$.

Theorem 6. $B(Q_{T_e}) \supset Cq_{T_e}$ and $B(B(Q_{T_e}))$ is the family of all functions with Baire property.

Remark 1.The results which are presented in the theorems 1-5 hold, if instead the basis $(\mathcal{P}, \Longrightarrow)$ we will use the basis of disc or squares, or all intervals.

Theorem 7. $A(Q_{T_e}) = Cq_{T_e}$.

An outline of proof.Let $f \in Cq_{T_e}$ .We have $f = g + h$, where $g,h \in Cq_{T_e}$ and for every $x \in R^m$ there exists a finite limit number $\alpha_g(x)$ of $g/C(g)$ and a finite limit number $\alpha_h(x)$ of $h/C(h)$. ($C(g)$ denotes the set of all continuity points of g). Define

$$m_1(x)=\begin{cases} g(x) & \text{if } g \text{ is continuous at } x \\ \alpha_g(x) & \text{if } g \text{ is not continuous at } x \end{cases}$$

$$m_2(x)=\begin{cases} h(x) & \text{if } h \text{ is continuous at } x \\ \alpha_h(x) & \text{if } h \text{ is not continuous at } x \end{cases}$$

$n_1 = g - m_1$ and $n_2 = h - m_2$.

Since m_1 and m_2 are T_e-quasi-continuous, it is enought to prove that $n_1 = \varphi_1 + \psi_1$ and $n_2 = \varphi_2 + \psi_2$, where $\varphi_1, \varphi_2, \psi_1, \psi_2$ are T_e-quasi-continuous.

Remark 2.The theorems 1-7 generalize more early results for real functions of one variable.

Let $R^m = R$. If $f:R \rightarrow R$ is a function, then denote by $Q(f)$ the set of all T_e-quasi-continuity points of f. Let Cq_o be the set $\{f \in Cq_{T_e} ; f:R \rightarrow R$ and $R-Q(f)$ is nowhere dense$\}$.

Theorem 8.If $R^m = R$, we have $L(Q_{T_e}) = Cq_o$.

Denote by d the density topology in R.

Theorem 9. Every d-continuous function $f: R \to R$ is a sum of two functions g,h which are d-continuous and T_e-quasi-continuous.

Theorem 10. Every derivative $f: R \to R$ is a sum of two T_e-quasi-continuous derivatives.

III. Problems. We have

(1) If each x section of a function $f: R^2 \to R$, $f_x(t) = f(x,t)$ and each y section $f^y(t) = f(t,y)$ are T_e-quasi-continuous, then f is T_e-quasi-continuous(Kempisty).

(2) There exists (under Martin Axiom) a function $f: R^2 \to R$ that all f_x and f^y are d-quasicontinuous, f is not$(d \times d)$-cliquish and f is not Lebesgue measurable.

(3) There exists a Lebesgue measurable function $f: R^2 \to R$ which is not $(d \times d)$-cliquish.

(4) If all f_x are d-continuous and if all f^y are d-quasi--continuous, then f is $(d \times d)$-cliquish.

(5) There exists a function $f: R^2 \to R$ such that all sections f_x and f^y are d-continuous and f is not $(d \times d)$-quasi-continuous.

Problem 1.Is any $(d \times d)$-quasi-continuous function $f: R^2 \to R$ Lebesgue measurable ?

O'Malley defines the following topology in R^2 :

$d_{xy} = \{A \subset R^2; A$ is measurable(L)and all sections $A_x, A^y \in d\}$.

A function $f: R^2 \to R$ is d_{xy}-cliquish iff it is measurable(L).

Problem 2.What is a characterization of family $Q_{d_{xy}}$?

Problem 3.Denote by r the O'Malley's topology r in R. What is a characterization of family Q_r ?