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John C. Morgan II, California State Polytechnic University,
Pomona, California 91768

ON THE GENERAL THEORY OF POINT SETS, II.

In the present article we state the most important open problems of the general theory and we indicate to what extent analogies between Baire category and Lebesgue measure are valid for other classifications of sets. Specifically, we shall enunciate general theorems true of all six primal examples discussed in [19], concerning Baire category, Lebesgue measure, Hausdorff measure, Hausdorff dimension, topological dimension, and Marczewski's classification.

We recall that the foundation of the general theory is the axiomatically defined notion of a category base. A pair $(X, \mathcal{C})$, where X is a nonempty set and $\mathcal{C}$ is a family of subsets of X, is called a category base if the nonempty sets in $\mathcal{C}$, called regions, satisfy the following axioms:

1. Every point of X belongs to some region; i. e. $X = \bigcup \mathcal{C}$
2. Suppose A is a region and $\mathcal{D}$ is a nonempty family of disjoint regions which has power less than the power of $\mathcal{C}$.
   a. If $A \cap (\bigcup \mathcal{D})$ contains a region then there is a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region.
   b. If $A \cap (\bigcup \mathcal{D})$ contains no region then there is a region $B \subset A$ which is disjoint from every region in $\mathcal{D}$.

With respect to a given category base $(X, \mathcal{C})$ the subsets of X are classified as follows: A set S is singular iff every region contains a subregion disjoint from S. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is called a Baire set iff every region contains a subregion whose intersection with either the set or its complement is meager.

We recall

(I) The singular sets form an ideal and the meager sets form a σ-ideal.

(II) The Baire sets form a σ-field containing all regions and all meager sets.

(III) The family of Baire sets is closed under operation $\mathcal{A}$.

In the case that $(X, \mathcal{C})$ is a topology, the singular, meager, and Baire sets are the nowhere dense sets, first category sets, and sets which have the Baire property (in the wide sense), respectively.

The following result due to K. Schilling (in a letter to the author) generalizes a theorem of Birkhoff and Ulam.

(IV) The quotient algebra of Baire sets modulo meager sets is a complete Boolean algebra.

We say a set S is meager (abundant) in a region A iff $S \cap A$ is meager (abundant). A set S is abundant everywhere in a region A iff S is abundant in every subregion of A. The fundamental theorem of the general theory of point sets is

(V) Any abundant set is abundant everywhere in some region.

The topological version of this theorem, known as the Banach Category Theorem, provided the break-through necessary to extend several theorems concerning Baire's category concepts from separable metric spaces to arbitrary topological spaces (cf. [1], [11], [15], [21], [28]). The generalization (V) of Banach's theorem enables the further extension of most of these theorems beyond topological spaces. It is surprising to observe that, in spite of its theoretical importance and the fact that it is virtually the only non-trival theorem true of every topological space, Banach's theorem is rarely encountered in modern textbooks on topology.

By a neighborhood of a point we understand any region containing the point. If Φ is a property of sets then we say a set has the property Φ locally at a given point iff every neighborhood of the point contains a neighborhood of the point in which the set has the property Φ. Concerning this notion of localization, we have for any category base

(VI) A set is meager if and only if it is locally meager at every point.

(VII) A set is a Baire set if and only if it is a Baire set locally at every point.

A category base is called a normal base iff every countable set is a meager set. For such bases one can use an Ulam matrix of sets to establish

(VIII) Every abundant set of power $\aleph_1$ can be decomposed into an uncountable family of disjoint abundant sets.

(IX) Every set of power $\aleph_1$ which is not a Baire set can be decomposed into an uncountable family of disjoint sets, none of which is a Baire set.

Having noted these basic properties we turn to formulate open problems, the most important of which is the

<u>PRODUCT PROBLEM</u>. Can the Kuratowski-Ulam theorem of topology and its measure analogue, Fubini's theorem, be unified within the context of the general theory?

We have shown in [20] that, for certain complete measures, the measurable sets coincide with the Baire sets for a suitable category base of measurable sets. For instance, if $(X, \mathfrak{A}, \mu)$ is the completion of a σ-finite measure structure $(X, \mathfrak{A}_0, \mu_0)$ then the family $\mathcal{C}$ of all μ_0-measurable sets of positive measure is a category base for which the singular and meager sets coincide with the sets of μ-measure zero and the Baire sets coincide with the μ-measurable sets. On the other hand, resolving the question raised in [20], K. Schilling (in a letter to the author) has shown that there are complete measure structures for which the measurable sets do not coincide with the Baire sets for any category base of measurable sets.

<u>MEASURABILITY PROBLEM</u>. Characterize those complete measure structures $(X, \mathfrak{A}, \mu)$ for which the measurable sets coincide with the Baire sets for a suitable category base of μ-measurable sets.

We recall that two category bases are called equivalent iff they yield the same meager sets and the same Baire sets. For example, the category base of all linear Borel sets of positive Lebesgue measure is equivalent to the density topology. In [18] it was shown that every finite category base is equivalent to a topology and the following problem, which remains unsolved, was posed.

<u>EQUIVALENCE PROBLEM</u>. Is every category base equivalent to a topology?

Our next problem concerns the extension of Sierpiński's duality theorem according to which there exists a one-to-one mapping f of the real line onto itself having the property that a set S is of the first category if and only if f(S) has Lebesgue measure zero (cf. [5], [22], [25]).

We call a set $S \subset \mathbb{R}$ an absolute null-set iff it has measure zero with respect to the completion of every continuous, σ-finite measure defined on the Borel sets in $\mathbb{R}$; or, equivalently, iff every homeomorphic image of S in $\mathbb{R}$ has Lebesgue measure zero.

DUALITY PROBLEM. Does there exist a one-to-one mapping f of the real line onto itself having the properties

(i) a set S is of the first category if and only if f(S) has Lebesgue measure zero,

(ii) a set S is always of the first category if and only if f(S) is an absolute null-set?

Of particular importance are the following category bases, which include bases equivalent to five of the category bases of [19]. All perfect sets considered below are assumed to be nonempty.

EXAMPLE A. $X = \mathbb{R}^n$ and $\mathcal{C}$ is the family of all closed rectangles. The singular, meager, and Baire sets coincide with the nowhere dense sets, first category sets, and sets with the Baire property (in the wide sense), respectively.

EXAMPLE B. $X = \mathbb{R}^n$ and $\mathcal{C}$ is the family of all perfect sets of positive Lebesgue measure in every open ball containing one of their points. The singular and meager sets both coincide with the sets of measure zero and the Baire sets are the Lebesgue measurable sets.

EXAMPLE C. $X = \mathbb{R}^2$ and $\mathcal{C}$ is the family of all product sets $A \times B$, where A and B are linear perfect sets of positive Lebesgue measure in every open interval containing one of their points.

EXAMPLE D. $X = \mathbb{R}^2$ and $\mathcal{C}$ is the family of all product sets $A \times B$, where A is a linear perfect set of positive Lebesgue measure in every open interval containing one of its points and B is a closed interval.

EXAMPLE E. $X = \mathbb{R}^n$ and $\mathcal{C}$ is the family of all perfect sets. This yields Marczewski's classification of [30].

A set singular for this classification necessarily contain no perfect sets. The first person to construct an uncountable set containing no perfect sets was G. H. Hardy (cf. [8], [9]), however he did not prove this fact. The proof that there exist uncountable sets which have no perfect subsets is due to Bernstein [2]. In [13] Luzin established the fact that Hardy's set is of the first category relative to every perfect set, from which it follows that it contains no perfect sets and is singular for the present classification.

EXAMPLE F. $X = \mathbb{R}^n$, L is a fixed line and $\mathcal{C}$ is the family of all closed line segments parallel to L.

EXAMPLE G. (Assume the Continuum Hypothesis). Let $X = \mathbb{R}^n$, let μ denote the Hausdorff measure associated with a monotone increasing, continuous function $h: [0,\infty) \longrightarrow [0,\infty)$ with $h(0) = 0$ and $h(t) > 0$ for $t > 0$, and let $\mathscr{C}$ be the family of all perfect sets of positive Hausdorff measure μ in every open ball containing one of their points. The singular and meager sets both coincide with the sets having no subset of finite, positive outer measure, while the Baire sets are identical to the μ-measurable sets.

If μ is not σ-finite then, assuming the Continuum Hypothesis, there exists an uncountable set in $\mathbb{R}^n$ every uncountable subset of which has infinite measure. Such a set is singular and has been so-termed by Choquet (cf. [3], [4], [20], [29]).

EXAMPLE H. (Assume the Continuum Hypothesis). Let $X = \mathbb{R}^n$, let p be a real number with $0 \leq p < n$, and let $\mathscr{C}$ be the family of all perfect sets of Hausdorff dimension larger than p in every open ball containing one of their points. For this example, a Borel set is meager if and only if it has Hausdorff dimension $\leq p$.

These category bases are instances of what we have called perfect bases (cf. [16], [17]). For such bases, in complete metric spaces X without isolated points, we have

(X) Every region is abundant and every countable set is meager.

(XI) Every analytic set is a Baire set.

(XII) Every abundant Baire set contains a perfect set.

(XIII) A set is meager if and only if every one of its subsets is a Baire set.

If the space X is also separable then, using Bernstein sets, one can show

(XIV) For each cardinal number $\mathfrak{m}$, with $2 \leq \mathfrak{m} \leq 2^{\aleph_0}$, the space X can be decomposed into $\mathfrak{m}$ disjoint sets, none of which is a Baire set.

(XV) The space X is representable as the union of more than $2^{\aleph_0}$ sets, the intersection of any different pair of which has power $< 2^{\aleph_0}$ and none of which is a Baire set.

Now, in order to extend the validity of these results to include the sixth classification of [19] we introduce the notion of an essentially perfect base.

Let X be a complete metric space with no isolated points. A category base $(X, \mathcal{C})$ is called an essentially perfect base iff there exists a subfamily $\mathcal{C}^*$ of $\mathcal{C}$ consisting of perfect sets and a family $\mathcal{H}$ of subsets of X such that

$$\mathcal{C} = \{A - M : A \in \mathcal{C}^* \text{ and } M \in \mathcal{H}_\sigma\}$$

and the following conditions hold :

(a) For every set $A \in \mathcal{C}^*$ and every point $x \in A$ there exists a descending sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of sets in $\mathcal{C}^*$ such that $x \in A_n$, $A_n \subset A$, and diam $(A_n) \leq \frac{1}{n}$ for each n .

(b) If T is any set in $\mathcal{H}$ then every region in $\mathcal{C}^*$ contains a subregion in $\mathcal{C}^*$ which is disjoint from T.

(c) If $A \in \mathcal{C}^*$, $F = B - N$ is a region in $\mathcal{C}$ of the given basic form, and $F \subset A$ then $B \subset A$.

The term "perfect base" is retained for the case where $\mathcal{C}^* = \mathcal{C}$ and $\mathcal{H} = \emptyset$, in which case conditions (b) and (c) are redundant.

The following two instances of essentially perfect bases are of particular interest, the second of which is equivalent to the sixth example of [19].

EXAMPLE I. (Assume the Continuum Hypothesis). $X = \mathbb{R}^n$ and $\mathcal{C}$ consists of all sets of the form A—M, where A is a closed line segment and M is a countable set. The non-measurable set of Sierpiński having at most two points on any line is singular for this category base, as well as for Example F (cf. [6],[23]).

EXAMPLE J. (Assume the Continuum Hypothesis). Let $X = \mathbb{R}^n$ or $X = \mathbb{R}^\infty$ (Hilbert space), let p be a non-negative integer smaller than the (topological) dimension of X, and let $\mathcal{C}$ consist of all sets of the form A—M, where A is a perfect set of finite dimension > p having the property that if G is any open ball containing a point of A then $A \cap G$ is not contained in an $\mathcal{F}_\sigma$-set of dimension $\leq$ p and M is an $\mathcal{F}_\sigma$-set of dimension $\leq$ p.

In [10] Hurewicz proved the existence of an uncountable set in $\mathbb{R}^\infty$, every uncountable subset of which has infinite dimension, is equivalent to the Continuum Hypothesis. Such a set is singular for the category base $(\mathbb{R}^\infty, \mathcal{C})$, for each p, and thus may be viewed as being "negligible".

We now give some properties of functions which have their origin in well-known classical theorems on Lebesgue measure. In addition to all the properties enumerated above, these properties are valid for every essentially perfect base $(X, \mathcal{C})$.

We assume throughout that Y is a separable metric space and all functions referred to are functions from X to Y.

A function f will be called a Baire function iff $f^{-1}(G)$ is a Baire set for every open set $G \subset Y$. In Example A the Baire functions coincide with the functions having the Baire property (in the wide sense). In Example B the Baire functions are the Lebesgue measurable functions.

First, we give a generalization of a weakened version of the so-called "Luzin theorem" for Lebesgue measurable functions (see[7] concerning the correct attribution of this theorem).

(XVI) If f is a Baire function then every abundant Baire set contains a perfect set on which f is continuous.

We note that the characteristic function of a Sierpiński set (i.e. an uncountable set every uncountable subset of which has positive Lebesgue outer measure; cf. [24]) is a non-measurable function satisfying the condition that in every Lebesgue measurable set of positive measure there is a perfect set on which the function is continuous.

The characteristic function of a Mahlo-Luzin set (i.e. an uncountable set every uncountable subset of which is of the second category; cf. [12], [14]) is a function which does not have the Baire property, but does satisfy the condition that in every second category set with the Baire property there is a perfect set on which the function is continuous.

In the case of Example E, yielding Marczewski's classification of sets, the converse of (XVI) is also true. For this category base, a function is a Baire function if and only if in every perfect set there is a perfect subset on which the function is continuous.

In connection with Example G we note that it is not in general true that if f is a Hausdorff measurable function then every measurable set of positive measure contains a perfect set on which f is continuous. It is however true that if f is a Hausdorff measurable function then every abundant measurable set contains a perfect set of positive measure on which f is continuous.

From the property (XVI) one can derive, as in [26],

(XVII) Assume X is also separable. Then there exists a function whose graph has a nonempty intersection with the graph of each Baire function.

Concerning sequences of functions we have

(XVIII) If a sequence of Baire functions converges pointwise to a function f then f is a Baire function.

(XIX) If a sequence of Baire functions converges pointwise to a function f then in every abundant Baire set there is a perfect set on which the sequence converges uniformly to f.

For Lebesgue measure, the well-known Egorov theorem yields a stronger conclusion than that given by (XIX). As shown in [22], the category analogue of Egorov's theorem is not true. Nevertheless, (XIX) does include a category analogue of a weakened version of Egorov's theorem.

(XX) If the iterated limit of a double sequence of Baire functions converges pointwise to a function f then for every abundant Baire set S we can find a perfect set $P \subset S$ and extract a single sequence from the double sequence which converges pointwise to f for all points of P.

This result includes a weakened version of a measure-theoretic theorem of Fréchet. As shown in [27], the category analogue of Fréchet's theorem also fails to hold. The statement (XX) does, however, include a category analogue of the weakened version of Fréchet's theorem.

We conclude this discussion with one further problem related to Cantor's unsolved problem of characterizing the sets of uniqueness for trigonometric series representations, which was the impetus behind his development of the basic topological concepts. A set $E \subset [0, 2\pi)$ is called a set of multiplicity iff there exists a trigonometric series converging to 0 for all points of $[0, 2\pi) - E$ whose coefficients are not all zero.

PROBLEM. Does the family of all closed sets of multiplicity form a category base?

Assuming the Continuum Hypothesis, this question reduces to that of determining whether every $\mathcal{G}_\delta$-set of multiplicity contains a closed set of multiplicity.

REFERENCES

1. S. Banach, Théorème sur les ensembles de première catégorie. Fund. Math. 16 (1930), 395-398.
2. F. Bernstein, Zur Theorie der trigonometrischen Reihen. Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Math.-Phys. Klasse, Sitzungsberichte 60 (1908), 325-338.
3. A. S. Besicovitch, Concentrated and rarified sets of points. Acta Math. 62 (1934), 289-300.
4. G. Choquet, Ensembles singuliers et structure des ensembles mesurables pour les mesures de Hausdorff. Bull. Soc. Math. France 74 (1946), 1-14.
5. P. Erdős, Some remarks on set theory. Annals of Math. (2) 44 (1943), 643-643.
6. B. R. Gelbaum & J. M. H. Olmsted, Counterexamples in Analysis. Holden-Day, San Francisco 1964. (esp. p. 142-144)
7. H. Hahn & A. Rosenthal, Set Functions. Univ. New Mexico Press, Albuquerque 1948. (esp. p. 148)
8. G. H. Hardy, A theorem concerning the infinite cardinal numbers. Quart. Jour. Math. 35 (1903), 87-94.
9. ___________, The continuum and the second number class. Proc. London Math. Soc. (2) 4 (1906), 10-17.
10. W. Hurewicz, Une remarque sur l'hypothèse du continu. Fund. Math. 19 (1932), 8-9.
11. K. Kuratowski. La propriété de Baire dans les espaces métriques. Fund. Math. 16 (1930), 390-394.
12. N. Luzin, Sur un problème de M. Baire. C. R. Acad. Sci. Paris 158 (1914), 1258-1261.
13. _________, Sur l'existence d'un ensemble non dénombrable qui est de première catégorie dans tout ensemble parfait. Fund. Math 2 (1921), 155-157.
14. P. Mahlo, Über Teilmengen des Kontinuums von dessen Mächtigkeit. Sitzungsberichte Sächs. Akad. Wissen. Leipzig, Mat.-Natur. Kl., 65 (1913), 283-315. (esp. Aufgabe 5)
15. E. Marczewski/Szpilrajn, Sur l'oeuvre scientifique de Stefan Banach, II. Théorie des fonctions réelles et théorie de la mesure. Colloq. Math. 1 (1948), 93-102. (esp. p. 96)

16. J. C. Morgan II, The absolute Baire property. Pacific Journ. Math. 65 (1976), 421-436.

17. ______________, On product bases. ibid. 99 (1982), 105-126.

18. ______________, On equivalent category bases. ibid. 105 (1983), 207-215.

19. ______________, On the general theory of point sets. Real Analysis Exchange 9 (1984), 345-353.

20. ______________, Measurability and the abstract Baire property. Rend. Circ. Mat. Palermo (2) 34 (1985), 234-244.

21. J. Oxtoby, The Banach-Mazur game and Banach category theorem. Contributions to the theory of games, Vol. 3. Annals of Math. Studies 39 (1957), 159-163.

22. ____________, Measure and Category, Springer Verlag, New York 1980 (2nd ed.).

23. W. Sierpiński, Sur un problème concernant les ensembles mesurables superficiellement. Fund. Math. 1 (1920), 112-115.

24. ____________, Sur l'hypothèse du continu ($2^{\aleph_0} = \aleph_1$). ibid. 5 (1924) 177-187. (esp. p. 184-185)

25. ____________, Sur la dualité entre la première catégorie et la mesure nulle. ibid. 22 (1934), 276-280.

26. ____________, Sur un problème concernant les fonctions mesurables. Ann. Sci. Univ. Jassy 24 (1938), 154-156.

27. ____________, Remarque sur les suites doubles des fonctions continues. Fund. Math. 32 (1939), 1-2.

28. A. H. Stone, Kernel constructions and Borel sets. Trans. Amer. Math. Soc. 107 (1963), 58-70.

29. E. Szpilrajn/Marczewski, Sur un ensemble non mesurable de M. Sierpiński. Spraw. Towarz. Nauk. Warszaw., Wyd. III, 24 (1931), 78-85.

30. ____________, Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles. Fund. Math. 24 (1935), 17-34.

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