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L-POINTS OF TYPICAL FUNCTIONS IN THE ZAHORSKI CLASSES

Functions considered in this paper will belong to the space $\mathcal{B}^1$, the space of Baire class one functions on the interval $[0,1]$ equipped with the metric of uniform convergence. Ever since Lebesgue [6], it has been known that any function in the space of bounded Baire class one functions on $[0,1]$, $b\mathcal{B}^1$, is the derivative of its indefinite integral except at a set of points which is both of measure zero and of first category. As in [4], for $f \in \mathcal{B}^1$, we call x an L-point of f if $\lim_{h\to 0} \frac{1}{h}\int_0^h f(x+t)dt = f(x)$, and we let

$$N(f) = \{x \in [0,1]: \lim_{h\to 0} \frac{1}{h}\int_0^h f(x+t)dt \text{ does not exist}\}.$$

Although the set of L-points for any function in $b\mathcal{B}^1$ is large in terms of measure and category, it was shown in [5] that for the typical (in the sense of category) function $f \in b\mathcal{B}^1$ the set $N(f)$ fails to be σ-porous. More specifically, it was shown in that paper that if $\mathcal{N} = \{f \in \mathcal{B}^1: N(f)$ is σ-porous$\}$, and if $\mathcal{F}$ is any of the spaces $\mathcal{B}^1$, $b\mathcal{B}^1$, $\mathcal{B}^1\mathcal{D}$ (the Baire one Darboux functions), or $b\mathcal{B}^1\mathcal{D}$, then $\mathcal{N} \cap \mathcal{F}$ is closed and nowhere dense in $\mathcal{F}$.

Thus, using Zahorski's [11] notation, we have the situation that the typical function in the space $b\mathcal{M}_0 = b\mathcal{M}_1 = b\mathcal{B}^1\mathcal{D}$ has a non-σ-porous set of points which fail to be L-points, while according to the classical result of Denjoy [3], every point of [0,1] is an L-point for every function in $b\mathcal{M}_5$, the space of bounded approximately continuous functions on [0,1]. Thus, it seems natural to inquire about the situation in the intermediate spaces $b\mathcal{M}_2$, $b\mathcal{M}_3$, and $b\mathcal{M}_4$, especially in light of the recent interesting results dealing with the behavior of typical functions in the Zahorski classes presented by Rinne in [8] and [9].

For the reader not familiar with the definitions of the Zahorski classes, we present them again here. Throughout we shall use $|E|$ to denote the Lebesgue measure of a measurable set E, $E \setminus S$ to represent the intersection of the set E with the complement of the set S, and χ_S to denote the characteristic function of the set S.

A set E is in class M_i if it is an $\mathcal{F}_\sigma$ set and:

$i = 0$ every x in E is a bilateral accumulation point of E

$i = 1$ every x in E is a bilateral condensation point of E

$i = 2$ for x in E and $\delta > 0$, $|(x - \delta,x) \cap E| > 0$ and $|(x,x + \delta) \cap E| > 0$

$i = 3$ for x in E and any sequence $\{I_n\}$ of intervals converging to x with $|I_n \cap E| = 0$ for all n, $\lim_{n\to\infty} |I_n|/\text{dist}(x,I_n) = 0$

$i = 4$ if there exists a sequence of closed sets K_n and a sequence of positive numbers r_n such that $E = \cup K_n$ and for every x in K_n and for every number $c > 0$ there is an $\epsilon(x,c) > 0$ such that $|E \cap (x + h,x + h + h_1)|/|h_1| > r_n$ for all h and h_1 satisfying $hh_1 > 0$, $h/h_1 < c$, and $|h + h_1| < \epsilon(x,c)$

$i = 5$ every x in E is a point of density of E.

A function f on [0,1] is in class $\mathcal{M}_i$ (i = 0,1,2,3,4,5) if each associated set is in class M_i. We then have $\mathcal{M}_0 = \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5$.

Properties of typical functions in the subspace usc of $\mathcal{B}^1$, consisting of the upper semi-continuous functions, have recently been investigated by Mustafa in [7]. (See also [2] by Ceder and Pearson.) We shall investigate the size of the set of L-points for functions in this and related classes as well. Indeed, the key to the present paper is the next theorem, wherein we construct a bounded, upper semi-continuous, $\mathcal{M}_4$ function f (i.e., $f \in \text{busc}\mathcal{M}_4$), having a given perfect set for its N(f).

THEOREM 1. *If P is any perfect set of measure zero in [0,1], there is a function $f \in \text{busc}\mathcal{M}_4$ on [0,1] such that (i) $P = N(f)$, and (ii) f is continuous at each $x \notin P$.*

Proof. Before beginning the construction of the function f, we introduce a sequence of $\text{busc}\mathcal{M}_4$ functions, $\beta_k : [0,1] \to [-2,1]$. To this end, for each natural number n, and for each integer i = 0, 1, ... 2^n-1, let

$$I_{n,i} = \left(\frac{1}{2^n} - \frac{i+1}{2^{2n+1}} + \frac{1}{2^{3n+2}}, \frac{1}{2^n} - \frac{i}{2^{2n+1}}\right)$$

and

$$J_{n,i} = \left(\frac{1}{2^n} - \frac{i+1}{2^{2n+1}}, \frac{1}{2^n} - \frac{i+1}{2^{2n+1}} + \frac{1}{2^{3n+2}}\right).$$

It is then easily verified that both of the sets

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^{n-1}-1} I_{n,2j} \quad \text{and} \quad \mathcal{I}^* = \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^{n-1}-1} I_{n,2j+1}$$

have right density one half at zero, and that the set

$$\mathcal{J} = \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{2^n-1} J_{n,i}$$

has right density zero at zero. Define the function $\beta : [0,1/2] \to [-2,1]$ by setting $\beta(x) = -2$ for $x \in \mathcal{I}$, $\beta(x) = 1$ for $x \in \mathcal{I}^*$, and then making β a linear function with range $[-2,1]$ on the closure of each $J_{n,i}$ in such a manner that the resulting function is continuous on $(0,1/2)$. Finally, set $\beta(0) = 1$, and $\beta(1/2) = -2$. Then β is continuous on $(0,1/2]$ and, as noted by Bruckner in [1, pp. 22,23,93], the density properties of the sets $\mathcal{I}$, $\mathcal{I}^*$, and $\mathcal{J}$ at zero are sufficient to readily conclude that β is in class $\mathcal{M}_4$, and it is clearly upper semicontinuous.

Now, for each natural number k, let $\beta_k : [0,1] \to [-2,1]$ be defined by

$$\beta_k(x) = \begin{cases} \beta(x/2^{k-1}), & \text{if } 0 \le x \le 1/2 \\ \beta((1-x)/2^{k-1}), & \text{if } 1/2 \le x \le 1. \end{cases}$$

Since β belongs to busc$\mathcal{M}_4$, so does each β_k. It is worth pausing at this point to take note of some of the properties of these β_k's. Fix a number h strictly between 0 and 1. It is then easily seen that both of the sequences $\{|\{x \in (0,h) : \beta_k(x) = 1\}|/h : k = 1,2, \ldots\}$ and $\{|\{x \in (h,1) : \beta_k(x) = 1\}|/(1-h) : k = 1,2, \ldots\}$ have limit 1/2 as $k \to \infty$. This, of course, follows from the earlier observation that the set $\mathcal{I}^*$ has right density 1/2 at 0. For future reference, we note that every term in both of these sequences is greater than 1/4, and we shall refer to this property of β_k as PROPERTY A. Now, fix a k. It is readily seen that for any positive constant $c < 2^{k-1}$, the following property, which we shall denote as PROPERTY B, holds: If h and h_1 are two positive numbers such that $\frac{h}{h_1} < c$

and $h + h_1 \in [0,1]$, then $\dfrac{|\{x: \beta_k(x) = 1\} \cap [h, h + h_1]|}{h_1} > \dfrac{1}{6}$. (The fact that PROPERTY B holds for larger and larger values of c as k increases is a feature that will be taken advantage of in the construction of the function f.) A third essential and easily verified feature of each β_k is that

$$\int_0^1 \beta_k(x)dx < -1/4,$$

and we shall refer to this inequality as PROPERTY C of β_k. Finally, if $x \notin [0,1]$, we agree to set $\beta_k(x) = 0$.

Let P be the given perfect set of measure zero in $I_0 = [0,1]$, and enumerate the component intervals of $I_0 \setminus P$ as a sequence $\{G_i = (a_i,b_i) : i = 1, 2, \ldots\}$. Choose N_1 so large that $|I_0 \cap (\bigcup_{i=1}^{N_1} G_i)| \geq 5/6$. In general, if N_k has been defined, let N_{k+1} be large enough to insure that if I is any component interval of $I_0 \setminus \bigcup_{i=1}^{N_k} G_i$, then $|I \cap \bigcap_{i=N_k+1}^{N_{k+1}} G_i| \geq 5|I|/6$. For notational purposes we shall set $N_0 = 0$. Then for each natural number k set $H_k = \bigcup_{i=N_{2k-2}+1}^{N_{2k-1}} G_i$, $H_k^* = \bigcup_{i=N_{2k-1}+1}^{N_{2k}} G_i$, $H = \bigcup_{k=1}^{\infty} H_k$, and $H^* = \bigcup_{k=1}^{\infty} H_k^*$. Then H and H^* are disjoint open sets with $H \cup H^* = I_0 \setminus P$. For each i we shall let k(i) denote that unique value of k for which either $G_i \subseteq H_{k(i)}$ or $G_i \subseteq H_{k(i)}^*$.

We now define our function f on [0,1] by

$$f(x) = \chi_P(x) + \sum_{k=1}^{\infty} \left[\chi_{H_k}(x) + \chi_{H_k^*}(x) \cdot \sum_{i=1}^{\infty} \beta_k\left[\frac{x - a_i}{b_i - a_i}\right] \right].$$

We shall first show that this function is in $\text{busc}\mathcal{M}_4$. It is clearly continuous at each point of $I_0 \setminus P$ and since the range of f is [-2,1] and f is identically 1 on P, it is obviously usc at each point of P. Suppose that $-2 \leq \alpha < 1$ (All other cases are immediate.) and consider the associated sets $E^{\alpha} = \{x: f(x) < \alpha\}$ and $E_{\alpha} = \{x: f(x) > \alpha\}$. Both are readily seen to be $\mathcal{F}_{\sigma}$ sets. Indeed, E^{α} is open and therefore an M_4 set. We must show that E_{α} is an M_4 set. Clearly, $P \subseteq E_{\alpha}$. Let $G = E_{\alpha} \setminus P$. It is easy to see that G is open. Indeed, if $x \in G$, then $x \in G_i$ for some unique i, and either $G_i \subseteq H_{k(i)}$ or $G_i \subseteq H^*_{k(i)}$. Consequently, f is continuous on the open interval G_i containing x, and, therefore, there is an open interval containing x which is completely contained in G. Consequently, we have E_{α} expressed as

$$E_{\alpha} = P \cup G, \tag{1}$$

where G is open and P is closed. Hence, to show that E_{α} is an M_4 set, it will suffice to produce a number, $r_1 > 0$, with the property that for each $x \in P$, and for every positive number c there is an $\epsilon(x,c) > 0$ such that $|E \cap (x + h, x + h + h_1)|/|h_1| > r_1$ for all h and h_1 satisfying $hh_1 > 0$, $h/h_1 < c$. and $|h + h_1| < \epsilon(x,c)$. Indeed, we shall show that $r_1 = 1/6$ will work.

Let $x \in P$. As a first case, suppose that x is a limit point from the right for P. Let c be any given positive number. Choose k so large that $2^{k-1} > c$, and then let $\epsilon = \epsilon(x,c)$ be so small that $(x, x+\epsilon) \subseteq I_0$ and the only G_i's which intersect $(x, x+\epsilon)$ have subscripts greater than N_{2k-2}. Now, suppose that h and h_1 are positive numbers satisfying $h/h_1 < c$, and $|h + h_1| < \epsilon$. We shall show that $|E_{\alpha} \cap (x + h, x + h + h_1)| > h_1/6$. Note that it will suffice to show that $|E \cap (x + h, x + h + h_1)| > h_1/6$, where $E = \{x: f(x) = 1\}$. We shall establish this latter inequality by considering the possible locations of the points $x + h$ and $x + h + h_1$.

Case I. Suppose that $x + h \in P$ and $x + h + h_1 \in P$. Let $\mathcal{L} = \{i: G_i \subseteq (x + h, x + h + h_1)\}$. For each $i \in \mathcal{L}$, either $G_i \subseteq H_{k(i)}$ or $G_i \subseteq H^*_{k(i)}$. If $G_i \subseteq H_{k(i)}$, then $G_i \subseteq E$, and if $G_i \subseteq H^*_{k(i)}$ then PROPERTY A of $\beta_{k(i)}$ assures that $|E \cap G_i| > |G_i|/4$. Consequently, $|E \cap (x + h, x + h + h_1)| = |E \cap \bigcup_{i\in\mathcal{L}} G_i| = |\bigcup_{i\in\mathcal{L}} (E \cap G_i)| = \sum_{i\in\mathcal{L}} |E \cap G_i| > \sum_{i\in\mathcal{L}} |G_i|/4 = h_1/4$.

Case II. Suppose that $x + h \notin P$ and $x + h + h_1 \notin P$. In this situation, there are two possibilities: $x + h$ and $x + h + h_1$ belong to the same G_i or they do not. Consider the former situation. Then $G_i \subseteq H_{k(i)}$ or $G_i \subseteq H^*_{k(i)}$. If $G_i \subseteq H_{k(i)}$, then $G_i \subseteq E$ and hence $|E \cap (x + h, x + h + h_1)| = h_1$. If $G_i \subseteq H^*_{k(i)}$, the selection of ϵ assures that $2^{k(i)-1} > c$, and since $0 < (x + h - a_i)/h_1 < h/h_1 < c$, we may apply PROPERTY B of $\beta_{k(i)}$ to obtain $|E \cap (x + h, x + h + h_1)| > h_1/6$. Consider now the latter situation where $x + h \in G_{i_o}$, $x + h + h_1 \in G_{j_o}$ and $i_o \neq j_o$. Letting $\mathcal{L} = \{i: b_{i_o} < a_i < b_i < a_{j_o}\}$ and applying the same reasoning as in *Case I*, we obtain $|E \cap (b_{i_o}, a_{j_o})| > \sum_{i\in\mathcal{L}} |G_i|/4 = (a_{j_o} - b_{i_o})/4$. Applying PROPERTY A of $\beta_{k(i_o)}$ and $\beta_{k(j_o)}$, we obtain $|E \cap (x + h, b_{i_o})| > (b_{i_o} - x - h)/4$ and $|E \cap (a_{j_o}, x + h + h_1)| > (x + h + h_1 - a_{j_o})/4$, respectively. Consequently, $|E \cap (x + h, x + h + h_1)| > h_1/4$.

Case III. Suppose that $x + h \notin P$ and $x + h + h_1 \in P$. Say $x + h \in G_{i_o} = (a_{i_o}, b_{i_o})$ and let $\mathcal{L} = \{i: b_{i_o} < a_i < b_i \leq x + h + h_1\}$. Either $G_{i_o} \subseteq E$ or PROPERTY A of $\beta_{k(i_o)}$ applies to yield $|E \cap (x + h, b_{i_o})| > (b_{i_o} - x - h)/4)$). Likewise, for each $i \in \mathcal{L}$, either $G_i \subseteq E$ or PROPERTY A of $\beta_{k(i)}$ applies to yield $|E \cap G_i| > |G_i|/4$. Consequently, $|E \cap (x + h, x + h + h_1)| > h_1/4$.

Case IV. The remaining case, where $x + h \in P$ and $x + h + h_1 \notin P$, can be handled in a manner similar to *Case III*.

We have shown in all cases that $|E \cap (x + h, x + h + h_1)| > h_1/6$.

Similarly, if x is assumed to be a limit point from the left of P, then a symmetric argument shows that for each positive c there is a positive $\epsilon(x,c)$ such that if h and h_1 are negative numbers satisfying $h/h_1 < c$, and $|h + h_1| < \epsilon$, then $|E \cap (x + h, x + h + h_1)| > |h_1|/6$.

On the other hand, if the point x in P is not a limit point from the right for P, then the situation is considerably simpler. For then $x = a_i$ for some $G_i = (a_i, b_i)$ and either $G_i \subseteq H_{k(i)}$ or $G_i \subseteq H^*_{k(i)}$. If $G_i \subseteq H_{k(i)}$, then $G_i \subseteq E$; and if $G_i \subseteq H^*_{k(i)}$, then since $f(t) = \beta_{k(i)}\left[\frac{t - a_i}{b_i - a_i}\right]$ for $t \in G_i$, we can, for any positive c, revert to the properties of the original function β to find an $\epsilon(x,c) > 0$ such that if h and h_1 are positive numbers satisfying $h/h_1 < c$, and $|h + h_1| < \epsilon$, then $|E \cap (x + h, x + h + h_1)| > h_1/6$. The symmetric situation holds in the case where the point $x \in P$ is not a limit point from the left for P. Consequently, for any $x \in P$ and any $c > 0$, by selecting $\epsilon(x,c)$ to be the minimum of those two candidates obtained as described above, depending upon whether x is a right (left) limit point or isolated from the right (left), we have that $|E \cap (x + h, x + h + h_1)|/|h_1| > 1/6$ for all h and h_1 satisfying $hh_1 > 0$, $h/h_1 < c$. and $|h + h_1| < \epsilon(x,c)$. This, together with (1), yields that E_α is an M_4 set, and, consequently, $f \in \text{busc}\mathcal{M}_4$.

As noted earlier, if $x \in I_o \setminus P$, then f is continous at x. Hence, it only remains to show that $P \subseteq N(f)$. Suppose $x \in P$. For each natural number k there is a component interval, call it I_k, of $I_o \setminus \bigcup_{i=1}^{N_k} G_i$ containing x.

If k is even, say $k = 2j - 2$ for some natural number j, then $|I_k \cap H_j| > 5|I_k|/6$, and consequently,

$$\frac{1}{|I_k|}\int_{I_k} f(t)dt = \frac{|I_k \cap H_j|}{|I_k|}\,\frac{1}{|I_k \cap H_j|}\int_{I_k \cap H_j} f(t)dt + \frac{1}{|I_k|}\int_{I_k \setminus H_j} f(t)dt$$

$$> \frac{5}{6}\cdot\frac{1}{|I_k \cap H_j|}\int_{I_k \cap H_j} 1dt - \frac{1}{|I_k|}\int_{I_k \setminus H_j} 2dt$$

$$> \frac{5}{6} - \frac{1}{3} = \frac{1}{2}\,.$$

On the other hand, if $k = 2j - 1$ for some natural number j, then $|I_k \cap H_j^*| > 5|I_k|/6$. If $\mathcal{L} = \{i\colon G_i \subseteq I_k \cap H_j^*\}$, then $I_k \cap H_j^* = \bigcup_{i \in \mathcal{L}} G_i$, and for each $i \in \mathcal{L}$ we have

$$\frac{1}{|G_i|}\int_{G_i} f(t)dt = \frac{1}{|G_i|}\int_{G_i} \beta_j\left[\frac{t - a_i}{b_i - a_i}\right]dt < -\frac{1}{4}\,,$$

where the inequality is a consequence of PROPERTY C of β_j. Thus we have

$$\frac{1}{|I_k \cap H_j^*|}\int_{I_k \cap H_j^*} f(t)dt < -\frac{1}{4}\,,$$

and hence

$$\frac{1}{|I_k|}\int_{I_k} f(t)dt = \frac{|I_k \cap H_j^*|}{|I_k|}\,\frac{1}{|I_k \cap H_j^*|}\int_{I_k \cap H_j^*} f(t)dt + \frac{1}{|I_k|}\int_{I_k \setminus H_j^*} f(t)dt$$

$$< -\frac{5}{6}\cdot\frac{1}{4} + \frac{1}{6} = -\frac{1}{24}\,.$$

Consequently, $x \in N(f)$, and the proof is complete.

The following simple observation, which is used in the proof of the next theorem, is quite probably well known, but since a reference is not known to us and since the proof is short, it is included.

REMARK *If f and g are* $\mathcal{M}_i$ *(i = 1,2,3,4, or 5) functions, and the sets of points of discontinuity of f and g are disjoint, then f + g is also an* $\mathcal{M}_i$ *function.*

PROOF. Let α be a real number and consider the associated set $E = \{x: f(x) + g(x) < \alpha\}$. Then $E = E(f) \cup E(g)$, where $E(f) = \{x \in E: f \text{ is continuous at } x\}$ and $E(g) = \{x \in E: g \text{ is continous at } x\}$. For each $x_o \in E(g)$, there is a rational number r such that $f(x_o) < r < \alpha - g(x_o)$. Since g is continuous at x_o there is an open interval I with rational endpoints containing x_o so that $g(x)$ is within $\epsilon = \alpha - r - g(x_o)$ of $g(x_o)$ on I. Then E contains the set $I \cap \{x|f(x) < r\}$, an M_i set. Similarly, if $x_o \in E(f)$, there is a rational number s such that $g(x_o) < r < \alpha - f(x_o)$. Since f is continuous at x_o there is an open interval J with rational endpoints containing x_o so that $f(x)$ is within $\epsilon = \alpha - s - f(x_o)$ of $f(x_o)$ on J. Then E contains the set $J \cap \{x|g(x) < s\}$, an M_i set. Thus E can be expressed as a countable union of M_i sets and therefore is an M_i set itself. The other associated set $\{x: f(x) + g(x) > \alpha\}$ can be handled similarly.

The proof of the next theorem is virtually identical to that of Theorem 2 in [5], but, again, since it is short, it is included.

THEOREM 2. *Let $\mathcal{F}$ be any of the following subsets of $\mathcal{B}^1$: $\mathcal{M}_i$ (i = 1,2,3,4), $b\mathcal{M}_i$ (i = 1,2,3,4), $usc\mathcal{M}_i$ (i = 1,2,3,4), $busc\mathcal{M}_i$ (i = 1,2,3,4), usc, busc. Then $\mathcal{N} \cap \mathcal{F}$ is a closed, nowhere dense subset of $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be any of the subspaces of $\mathcal{B}^1$ listed in the theorem statement. From the lemma in [5], we have that $\mathcal{N} \cap \mathcal{F}$ is closed in $\mathcal{F}$. Let $g \in \mathcal{N} \cap \mathcal{F}$ and let ϵ denote an arbitrary positive number. Let C(g) be the dense $\mathcal{G}_\delta$ set consisting of the continuity points of g. According to Theorem 2 in [10], there is a perfect, non-σ-porous set P of measure zero contained in C(g). Let f be the function from Theorem 1 of the current paper constructed using this set P. Let $h(x) = g(x) + \epsilon f(x)$. Regardless which of the various possibilities that $\mathcal{F}$ may be, the fact that the points of discontinuity of f and g are disjoint assure that $h \in \mathcal{F}$. It is then an easy matter to verify that $N(h) \supseteq N(f) = P$, implying that N(h) is non-σ-porous, and, hence, that $h \in \mathcal{F} \setminus \mathcal{N}$. Consequently, $\mathcal{N} \cap \mathcal{F}$ is nowhere dense in $\mathcal{F}$.

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