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LENGTHS OF RECTIFIABLE CURVES IN 2-SPACE

Let MC denote the family of nondecreasing continuous functions on [0,1], and let BC denote the family of continuous functions of bounded variation on [0,1]. Throughout this paper $(g(t), f(t))$ $(0 \le t \le 1)$ denotes a continuous rectifiable curve in R^2, i.e., $f,g \in BC$. We propose to determine the length L of this curve in terms of the functions f and g. A well-known result [5, p. 123] is

Proposition 1. We have $L \ge \int_0^1 ((f')^2 + (g')^2)^{1/2}$, and equality holds if and only if f and g are absolutely continuous on [0,1].

We want to express L in terms of f and g in a more general setting. To this end, we introduce a notation from [3]. If A is any subset of [0,1], measurable or not, let

$$M(F,A) = \lim_{m\to\infty} \sum_{i=1}^{2^m} \lambda F(J_{im} \cap A)$$

where λ is Lebesgue outer measure and $J_{im} = [(i-1)2^{-m}, i2^{-m}]$. Note that the expression after the limit increases with m. Moreover, $M(F,A) \le V(F)$, the total variation of F on [0,1]. If F is monotonic, clearly $M(F,A) = \lambda F(A)$. Also $M(F,A) = 0$ if and only if $\lambda F(A) = 0$.

Let $E_F = \{x : F'(x) = \pm\infty\}$. We need the

Definition: We say that f is <u>compatible</u> with g if there exist sets S_f and S_g such that $S_f \cup S_g = E_f \cap E_g$ and $\lambda f(S_f) = \lambda g(S_g) = 0$.

We offer

Theorem 1. Let $f, g \in BC$ and let L be the length of the curve $(g(t), f(t))$ $(0 \leq t \leq 1)$. Then

$$(*) \qquad L \leq \int_0^1 ((f')^2 + (g')^2)^{1/2} + M(f, E_f) + M(g, E_g),$$

and equality holds in $(*)$ if and only if f is compatible with g. Moreover, equality holds in $(*)$ if the set $E_f \cap E_g$ is at most countable.

When f is not compatible with g, we offer no further equations for L. Note that if $h \in MC$ and $h(0) = 0$, $h(1) = 1$, then no matter how complicated the function h is, the curve $(h(t), h(t))$ $(0 \leq t \leq 1)$, is the line segment joining $(0,0)$ to $(1,1)$.

It is obvious that $L \leq V(f) + V(g)$. We identify the extreme situation in which equality holds here.

Theorem 2. We have

$$(**) \qquad L \leq V(f) + V(g).$$

Moreover, equality holds in $(**)$ if and only if $f'g' = 0$ almost everywhere on $[0,1]$ and f is compatible with g.

We also identify another extreme situation.

Theorem 3. We have

$$(***) \qquad L \geq \int_0^1 ((f')^2 + (g')^2)^{1/2} + M(g, E_g),$$

and equality holds in $(***)$ if and only if f is absolutely continuous on $[0,1]$.

Before we tackle the proofs of Theorems 1, 2 and 3, we show the connection between absolute continuity and compatibility.

Proposition 2. Let $f \in BC$. Then the following are equivalent

(a) f is absolutely continuous on $[0,1]$.

(b) f is compatible with f.

(c) f is compatible with every function in BC.

Proof. (c) => (b). Clear.

Proof. (b) => (a). From (b) it follows that $\lambda f(E_f) = 0$. Then for any set $B \subset [0,1]$ satisfying $\lambda(B) = 0$, we have $\lambda f(B) = \lambda f(B \setminus E_f)$, and by [5, p. 271], $\lambda f(B \setminus E_f) = 0$. Thus f maps sets B of measure 0 to sets of measure 0, so f is absolutely continuous.

Proof. (a) => (c). For any $g \in$ BC, let $S_f = E_f \cap E_g$. Since $\lambda(S_f) = 0$ and f is absolutely continuous, we have $\lambda f(S_f) = 0$. Then f is compatible with g. □

Thus equality holds in Proposition 1 if f and g are compatible with all functions in BC, but equality holds in Theorem 1 if f and g are compatible with each other.

We first prove Theorem 1 in a very special case.

Lemma 1. Let $f,g \in$ MC such that $f' = g' = 0$ almost everywhere on $[0,1]$. Then $L \leq f(1) - g(0) + f(1) - f(0)$, and equality holds if and only if f is compatible with g.

Proof. The inequality $L \leq g(1) - g(0) + f(1) - f(0)$ is evident from the triangle inequality and the definition of L.

Now let $L = g(1) - g(0) + f(1) - f(0)$. Without loss of generality we assume that $g(1) > g(0)$ and $f(1) > f(0)$. Choose any ε, $0 < \varepsilon < 1$. Let $0 = u_0 < u_1 < u_2 < \cdots < u_n = 1$ be a partition of $[0,1]$ so fine that, setting $a_j = f(u_j) - f(u_{j-1})$, $b_j = g(u_j) - g(u_{j-1})$, $c_j = (b_j^2 + a_j^2)^{1/2}$, we have

$$V(g) - \sum_{j=1}^{n} b_j < \varepsilon, \qquad V(f) - \sum_{j=1}^{n} a_j < \varepsilon, \qquad \text{and}$$

(1)
$$L - \sum_{j=1}^{n} c_j < \tfrac{1}{2}\varepsilon^2.$$

But if

$$(2) \qquad \varepsilon\, a_j \leq b_j \leq a_j/\varepsilon,$$

then $(a_j+b_j)^2 \leq a_jb_j(2 + 2/\varepsilon) \leq 4a_jb_j/\varepsilon$ so that

$$2(a_j+b_j)(a_j+b_j-c_j) \geq (a_j+b_j+c_j)(a_j+b_j-c_j)$$

$$= 2a_jb_j \geq \varepsilon(a_j+b_j)^2/2,$$

$$a_j+b_j-c_j \geq \varepsilon(a_j+b_j)/4.$$

Now it follows from (1) that $\frac{1}{4}\varepsilon\, \Sigma_*(a_j+b_j) < \frac{1}{2}\varepsilon^2$, where Σ_* means the sum over those j that satisfy (2). Hence $\Sigma_*(a_j+b_j) < 2\varepsilon$.

Note that if j does not satisfy (2), then

$$|a_j-b_j| \geq a_j+b_j - 2\varepsilon \max(a_j,b_j).$$

Let Σ_{**} mean the sum over those j not satisfying (2); thus $\sum_{j=1}^{n} = \Sigma_* + \Sigma_{**}$. Then

$$\sum_{j=1}^{n} |a_j-b_j| > -2\varepsilon - 2\varepsilon\, \Sigma_{**} \max(a_j,b_j) + \Sigma_{**}(a_j+b_j)$$

$$> -4\varepsilon - 2\varepsilon(f(1) - f(0) + g(1) - g(0)) + \sum_{j=1}^{n} (a_j+b_j)$$

$$= -\varepsilon(4 + 2(f(1) - f(0) + g(1) - g(0)))$$

$$+ f(1) - f(0) + g(1) - g(0).$$

It follows that $V(f-g) \geq f(1) - f(0) + g(1) - g(0) = V(f) + V(-g)$. By [1], there exist sets S_f and S_g such that $E_f \cap E_g \subset S_f \cup S_g$, and $\lambda f(S_f) = \lambda g(S_g) = 0$. Hence f is compatible with g.

Now let f be compatible with g. Let S_f and S_g be the sets for which $S_f \cup S_g = E_f \cap E_g$ and $\lambda f(S_f) = \lambda g(S_g) = 0$. Let

$A = \{x : f$ has no finite or infinite derivative at $x\}$,

$A_0 = \{x : f$ is differentiable at x and $0 < f'(x) < \infty\}$,

$B = \{x : g$ has no finite or infinite derivative at $x\}$,

$B_0 = \{x : g$ is differentiable at x and $0 < g'(x) < \infty\}$.

Then $\lambda A_0 = \lambda B_0 = 0$ because $f' = g' = 0$ almost everywhere. By [5, pp. 125, 271] $\lambda f(A) = \lambda f(A_0) = \lambda g(B) = \lambda g(B_0) = 0$ and $\lambda f(A \cup A_0 \cup S_f) = \lambda g(B \cup B_0 \cup S_g) = 0$. But all the points where both f and g have positive derived numbers are in the set $(A \cup A_0 \cup S_f) \cup (B \cup B_0 \cup S_g)$. By [1], $V(f-g) = V(f) + V(-g)$. By the triangle inequality and the definition of L, we have

$$V(f-g) \leq L \leq V(f) + V(-g) = f(1) - f(0) + g(1) - g(0),$$

and equality must hold throughout. □

Our next lemma will eventually allow us to generalize Lemma 1 to all functions in MC.

Lemma 2. Let f_1 and g_1 be absolutely continuous functions on $[0,1]$ in MC, and let f_2 and g_2 be functions in MC such that $f_2' = g_2' = 0$ almost everywhere on $[0,1]$. Let $f = f_1 + f_2$, $g = g_1 + g_2$. Let L_j = length of the curve $(g_j(t), f_j(t))$ $(0 \leq t \leq 1)$, $(j = 1,2)$, L = length of the curve $(g(t), f(t))$ $(0 \leq t \leq 1)$. Then

(i) $L = L_1 + L_2$.

(ii) If $A \subset [0,1]$, then $\lambda f(A) = \lambda f_1(A) + \lambda f_2(A)$, $\lambda g(A) = \lambda g_1(A) + \lambda g_2(A)$.

(iii) $\lambda f(E_f \setminus A_f) = \lambda f_2(E_f \setminus A_f) = 0$ and $A_f \subset E_f$,
$\lambda g(E_g \setminus A_g) = \lambda g_2(E_g \setminus A_g) = 0$ and $A_g \subset E_g$,
where $A_f = \{x : f_2'(x) = \infty\}$, $A_g = \{x : g_2'(x) = \infty\}$.

Proof (i). From the triangle inequality and the definition of length we obtain $L \leq L_1 + L_2$ and $L \geq |L_1 - L_2|$. Take any $\varepsilon > 0$. There is a $\delta > 0$ such that if $\lambda(S) < \delta$, then $\int_S((f_1')^2 + (g_1')^2)^{1/2} < \varepsilon$. Since $f_2' = g_2' = 0$ almost everywhere, we can (and do) use the Vitali covering theorem to construct mutually disjoint subintervals $[u_j,v_j]$ $(j = 1,\dots,n)$ of $[0,1]$ such that

$$\sum_{j=1}^{n} (f_2(v_j) - f_2(u_j)) < \varepsilon, \quad \sum_{j=1}^{n} (g_2(v_j) - g_2(u_j)) < \varepsilon, \quad 1 - \sum_{j=1}^{n} (v_j-u_j) < \delta.$$

Let (x_j,y_j) $(j = 1,\dots,m)$ denote the complementary intervals of the $[u_j,v_j]$. Then $\sum_{j=1}^{m} (y_j-x_j) < \delta$.

Let Z_j denote the length of the curve $(g(t),f(t))$ for $u_j \le t \le v_j$, let Z_{1j} denote the length of the curve $(g_1(t),f_1(t))$ for $u_j \le t \le v_j$, and let Z_{2j} denote the length of the curve $(g_2(t),f_2(t))$ for $u_j \le t \le v_j$. Let Z_j^*, Z_{1j}^* and Z_{2j}^* denote the corresponding lengths for $x_j \le t \le y_j$. Then

$$\sum_j Z_{2j} \le \sum_j(f_2(v_j) - f_2(u_j)) + \sum_j(g_2(v_j) - g_2(u_j)) < 2\varepsilon,$$

$$\sum_j Z_{1j}^* \le \sum_j \int_{x_j}^{y_j} ((f_1')^2 + (g_1')^2)^{1/2} = \int_{\cup_j[x_j,y_j]} ((f_1')^2 + (g_1')^2)^{1/2} < \varepsilon$$

because $\lambda(\cup_j[x_j,y_j]) < \delta$. So

$$\begin{aligned} L = \sum_j Z_j + \sum_j Z_j^* &\ge \sum_j (Z_{1j}-Z_{2j}) + \sum_j (Z_{2j}^* - Z_{1j}^*) \\ &\ge \sum_j (Z_{1j}+Z_{2j}) - 4\varepsilon + \sum_j (Z_{2j}^*+Z_{1j}^*) - 4\varepsilon \\ &= L_1 + L_2 - 8\varepsilon. \end{aligned}$$

Since ε was arbitrary, $L \ge L_1 + L_2$. The reverse inequality yields (i).

Proof (ii). This is just [2, Lemma 3].

Proof (iii). Since f_1 and g_1 are nondecreasing, $f_2'(x) = \infty$ implies $f'(x) = \infty$, and $g_2'(x) = \infty$ implies $g'(x) = \infty$. Thus $A_f \subset E_f$ and $A_g \subset E_g$. Now $\lambda(E_f) = 0$, so $\lambda(E_f \setminus A_f) = 0$. It follows from [5, p. 271] that $\lambda f_2(E_f \setminus A_f) = 0$. But

$$\lambda f(E_f\setminus A_f) = \lambda f_1(E_f\setminus A_f) + \lambda f_2(E_f\setminus A_f) = \lambda f_2(E_f\setminus A_f) = 0$$

because f_1 is absolutely continuous. Likewise $\lambda g(E_g \backslash A_g) = \lambda g_2(E_g \backslash A_g) = 0$.

□

We are now able to prove Theorem 1 for functions in MC.

Lemma 3. Let f and g be functions in MC. Then

$$(*) \qquad L \le \int_0^1 ((f')^2 + (g')^2)^{1/2} + \lambda f(E_f) + \lambda g(E_g),$$

and equality holds if and only if f is compatible with g.

Proof. Let $f = f_1 + f_2$, $g = g_1 + g_2$, where $f_j \in MC$, $g_j \in MC$, f_1 and g_1 are absolutely continuous on $[0,1]$, and $f_2' = g_2' = 0$ almost everywhere on $[0,1]$. By Lemma 2, $\lambda f(E_f \backslash A_f) = 0$ and $\lambda f(E_f) = \lambda f(A_f) = \lambda f_2(A_f) = f_2(1) - f_2(0)$. Likewise $\lambda g(E_g) = g_2(1) - g_2(0)$. Also $f' = f_1'$ and $g' = g_1'$ almost everywhere on $[0,1]$. Using Lemmas 1 and 2 we obtain $(*)$ and we see that equality holds there if and only if f_2 is compatible with g_2.

Now suppose equality holds in $(*)$. Then f_2 is compatible with g_2. Let S_f and S_g be sets such that $A_f \cap A_g \subset S_f \cup S_g$, $\lambda(S_f \cup S_g) = 0$, and $\lambda f_2(S_f) = \lambda g_2(S_g) = 0$. But by Lemma 2, $\lambda f(S_f) = \lambda f_2(S_f) = 0$ and $\lambda f(E_f \backslash A_f) = 0$, and hence $\lambda f((E_f \backslash A_f) \cup S_f) = 0$. Likewise $\lambda g((E_g \backslash A_g) \cup S_g) = 0$. Thus f is compatible with g because $E_f \cap E_g \subset ((E_f \backslash A_f) \cup S_f) \cup ((E_g \backslash A_g) \cup S_g)$.

Suppose f is compatible with g. Let S_3 and S_4 be sets with $S_3 \cup S_4 = E_f \cap E_g$ and $\lambda f(S_3) = \lambda g(S_4) = 0$. But $\lambda f(S_3) = \lambda f_1(S_3) + \lambda f_2(S_3)$ so $\lambda f_2(S_3) = 0$. Likewise $\lambda g_2(S_4) = 0$. Finally, $A_f \cap A_g \subset E_f \cap E_g = S_3 \cup S_4$, and it follows that f_2 is compatible with g_2. Hence equality holds in $(*)$. □

Our next lemma will help us to prove Theorem 1 for functions in BC as well as functions in MC.

Lemma 4. Let $f, g \in BC$ and let $f_*(x)$ and $g_*(x)$ denote, respectively, the total variations of f and g on the interval $[0,x]$ $(0 \le x \le 1)$. Let

L denote the length of the curve $(g(t), f(t))$ and let L_* denote the length of the curve $(g_*(t), f_*(t))$ $(0 \leq t \leq 1)$. Then $L = L_*$.

Proof. Take any $\varepsilon > 0$. Let $0 = u_0 < u_1 < \cdots < u_n = 1$ be a partition of $[0,1]$ so fine that, setting

$$a_j = |f(u_j) - f(u_{j-1})|, \qquad b_j = |g(u_j) - g(u_{j-1})|,$$

$$a_{*j} = f_*(u_j) - f_*(u_{j-1}), \qquad b_{*j} = g_*(u_j) - g_*(u_{j-1}),$$

$$c_j = (a_j^2 + b_j^2)^{1/2}, \qquad c_{*j} = (a_{*j}^2 + b_{*j}^2)^{1/2},$$

we have

(1) $$L_* - \sum_{j=1}^{n} c_{*j} < \varepsilon,$$

(2) $$f_*(1) - f_*(0) - \sum_{j=1}^{n} a_j < \varepsilon,$$

(3) $$g_*(1) - g_*(0) - \sum_{j=1}^{n} b_j < \varepsilon.$$

It follows from (2) and (3) that

(4) $$\sum_{j=1}^{n} (a_{*j} - a_j) < \varepsilon,$$

(5) $$\sum_{j=1}^{n} (b_{*j} - b_j) < \varepsilon.$$

From the triangle inequality and (4), (5) we obtain

(6) $$L \geq \sum_{j=1}^{n} c_j \geq \sum_{j=1}^{n} (c_{*j} - (b_{*j} - b_j) - (a_{*j} - a_j)) \geq \sum_{j=1}^{n} c_{*j} - 2\varepsilon.$$

By (1) and (6) we have $L \geq L_* - 3\varepsilon$. Since ε is arbitrary, $L \geq L_*$. The reverse inequality follows from $a_{*j} \geq a_j$ and $b_{*j} \geq b_j$. □

We are now ready to prove Theorems 1 and 2.

Proof of Theorem 1. Define f_* and g_* as in Lemma 4. Then by Lemmas 3 and 4,

$$(*) \qquad L \le \int_0^1 ((f_*')^2 + (g_*')^2)^{1/2} + \lambda f_*(E_{f_*}) + \lambda g_*(E_{g_*}),$$

and equality holds in (*) if and only if f_* is compatible with g_*. Also $E_f \subset E_{f_*}$ and $E_g \subset E_{g_*}$ are clear, and by [5, p. 127], we have

$$\lambda(E_{f_*} \setminus E_f) = \lambda f_*(E_{f_*} \setminus E_f) = 0, \quad \lambda(E_{g_*} \setminus E_g) = \lambda g_*(E_{g_*} \setminus E_g) = 0,$$

$$\lambda f_*(E_f) = \lambda f_*(E_{f_*}), \quad \lambda g_*(E_g) = \lambda g_*(E_{g_*}).$$

Moreover $\lambda f(S) = 0$ for any set S with $\lambda f_*(S) = 0$ by [3, Lemma 2]. Likewise, $\lambda g(S) = 0$ for any set S with $\lambda g_*(S) = 0$. It follows that if f_* is compatible with g_*, so must f be compatible with g.

Now suppose f is compatible with g. Say $S_f \cup S_g = E_f \cap E_g$ and $\lambda f(S_f) = \lambda g(S_g) = 0$. Then $E_{f_*} \cap E_{g_*} \subset ((E_{f_*} \setminus E_f) \cup S_f) \cup ((E_{g_*} \setminus E_g) \cup S_g)$ and by [3, Lemma 2],

$$\lambda f_*((E_{f_*} \setminus E_f) \cup S_f) = \lambda g_*((E_{g_*} \setminus E_g) \cup S_g) = 0.$$

Thus f_* is compatible with g_*.

But $\lambda f_*(E_f) = M(f, E_f)$ and $\lambda g_*(E) = M(g, E_g)$ by [3, Lemma 2]. From this, (*) and from the fact that $f_*' = |f'|$, $g_*' = |g'|$ almost everywhere, it follows that

$$L \le \int_0^1 ((f')^2 + (g')^2)^{1/2} + M(f, E_f) + M(g, E_g),$$

and equality holds if and only if f is compatible with g.

The last statement in Theorem 1 is now clear. □

Note that when g is constant, then the total variation, $V(f)$, of f on $[0,1]$ equals L. But then f is compatible with g and

$$L = \int_0^1 |f'| + M(f,E_f) = V(f).$$

Proof of Theorem 2. The inequality (**) follows from the definition of L and the triangle inequality.

Now suppose $L = V(f) + V(g)$. Then by the remark preceding this proof, we have

$$L = V(f) + V(g) = \int_0^1 |f'| + M(f,E_f) + \int_0^1 |g'| + M(g,E_g)$$
$$\geq \int_0^1 ((f')^2 + (g')^2)^{1/2} + M(f,E_f) + M(g,E_g).$$

Clearly equality must hold throughout, and by Theorem 1, f is compatible with g. From $\int_0^1 (|f'| + |g'|) = \int_0^1 ((f')^2 + (g')^2)^{1/2}$ we obtain $f'g' = 0$ almost everywhere on $[0,1]$.

Suppose f is compatible with g and $f'g' = 0$ almost everywhere on $[0,1]$. Then by Theorem 1

$$L = \int_0^1 ((f')^2 + (g')^2)^{1/2} + M(f,E_f) + M(g,E_g)$$
$$= \int_0^1 (|f'| + |g'|) + M(f,E_f) + M(g,E_g) = V(f) + V(g).$$

□

Before we consider Theorem 3 we note some corollaries of Theorem 2.

Corollary 1. In Theorem 2, let $g' = 0$ almost everywhere on $[0,1]$. Then $L = V(f) + V(g)$ if and only if f is compatible with g.

Proof. $f'g' = 0$ almost everywhere on $[0,1]$ in any case. □

Corollary 2. In Theorem 2, let g be absolutely continuous on $[0,1]$. Then $L = V(f) + V(g)$ if and only if $f'g' = 0$ almost everyhwere on $[0,1]$.

Proof. Since g is absolutely continuous, $\lambda g(E_f \cap E_g) = 0$ and f is compatible with g in any case. □

Corollary 3. In Theorem 2, let g have a finite nonzero derivative everywhere on $[0,1]$ except possibly at countably many points. Then $L = V(f) + V(g)$ if and only if $f' = 0$ almost everywhere on $[0,1]$.

Proof. Here E_g is countable, so f is compatible with g. The rest is clear. □

From Corollary 3, we see that the curve $y = f(x)$, $0 \leq x \leq 1$, has length $= 1 + V(f)$ if and only if $f' = 0$ almost everywhere on $[0,1]$. This equation holds, for example, when f is Lebesgue's singular function [4, p. 113].

Proof of Theorem 3. The development of Theorem 3 is much like the development of Theorem 1, so we only sketch the procedure. First suppose f and g satisfy the hypothesis of Lemma 1. Then (***) reduces to $L \geq g(1) - g(0)$, and equality holds if and only if f is constant. This is obvious.

Now suppose f and g are as in Lemma 3. Then (***) reduces to

$$L \geq \int_0^1 ((f')^2 + (g')^2)^{1/2} + \lambda g(E_g),$$

and equality holds if and only if f_2 is constant, or equivalently, f is absolutely continuous on $[0,1]$. This follows from Lemmas 1 and 2, just as Lemma 3 did.

The proof of Theorem 3 is now analogous to the proof of Theorem 1, only it is easier. In the notation used in the proof of Theorem 1, we have

$$L \geq \int_0^1 ((f_*')^2 + (g_*')^2)^{1/2} + \lambda g_*(E_{g_*}),$$

and equality holds if and only if f_* is absolutely continuous, or equivalently, f is absolutely continuous on $[0,1]$. But as before, $((f_*')^2 + (g_*')^2)^{1/2} = ((f')^2 + (g')^2)^{1/2}$ almost everywhere on $[0,1]$, and

$\lambda g_*(E_{g_*}) = \lambda g_*(E_g) = M(g,E_g)$. The conclusion follows. □

Much of this work can be generalized to continuous rectifiable curves $(f_1(t),\ldots,f_n(t))$ $(0 \le t \le 1)$ in R^n. For example,

$$L \le \int_0^1 (\sum_{j=1}^{n} (f_j')^2)^{\frac{1}{2}} + \sum_{j=1}^{n} M(f_j,E_{f_j}),$$

and equality holds if and only if f_i is compatible with f_j for $i \ne j$. but the proof is a tedious induction argument that would add little original to the arguments here. So we omit it.

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