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Ibrahim Mustafa, 5779 Quicksilver Cir, Las Vegas, NV 89110

A General Approach Leading To Typical Results

Introduction. Notations. In this paper, we show that if Φ is a closed subfamily of the bounded Darboux Baire 1 functions, and if Φ is closed with respect to the addition of a continuous, piecewise linear function, then many of the properties known to be typical in bounded Darboux Baire 1 are also typical in Φ.

We shall see, in Lemma A2, that the subfamilies of bounded Darboux Baire 1 functions satisfying the above conditions include the families of continuous functions, bounded Darboux upper semi-continuous functions, bounded Darboux lower semi-continuous functions, bounded derivatives, and the bounded Zahorski classes. These families will be denoted by $\mathcal{C}$, b$\mathcal{D}$usc, b$\mathcal{D}$lsc, bΔ, and b$\mathcal{M}_i$ $(i=1,2,..,5)$, respectively. Note that b$\mathcal{M}_1$=b$\mathcal{DB}_1$ ([10]), we will use either notation for this class. Various properties have been shown to be typical in some of these families, see [3], [4], [5], [6], [7], [8], and [9].

Throughout, we assume that all functions are defined on the closed unit interval [0,1], which is denoted by I. Each of the above mentioned families is a Banach space with norm $||f||=\sup|f|$. For any function f, Gr(f) and C(f) denote, respectively, the graph of f and the continuity points of f. For any set A, $f|A$ denotes the restriction of f to A. The closure and interior of A are denoted by clA and IntA, respectively . If A is a nonempty subset of the plane then, domA denotes $\{x:(x,y)\in A\}$. Finally, R will denote the real numbers.

A subfamily Φ of $b\mathcal{DB}_1$ will be called an L-family, if it is closed in $b\mathcal{DB}_1$, and whenever f is in Φ and p is a real-valued, continuous, piecewise linear function defined on I, then f+p is in Φ.

In the following, Φ will denote an arbitrary L-family unless we explicitly state otherwise.

A. **Preliminary Results.** In this section we prove Lemma A2 which was mentioned in the introduction. First, we state Lemma A1 which is needed in its proof.

Lemma A1. If $f\in b\mathcal{DB}_1$ and $g\in\mathcal{C}$, then $f+g\in b\mathcal{DB}_1$.

Proof. [2] Theorem 3.2.

Lemma A2. Each of the families $\mathcal{C}$, $b\mathcal{D}$usc, $b\mathcal{D}$lsc, $b\Delta$, and $b\mathcal{M}_i$ $(i=1,2,..,5)$ is an L-family.

Proof. Each of the families above is closed in $b\mathcal{DB}_1$. (See [2] and [9].) Let p be a real-valued, continuous, piecewise linear function defined on I. By Lemma A1, $\Phi+p \subset b\mathcal{DB}_1$ for any family Φ appearing in the statement of this lemma. Moreover, it is clear that $\Phi+p \subset \Phi$ whenever Φ is one of $\mathcal{C}$, $b\mathcal{D}$usc, $b\mathcal{D}$lsc, or $b\Delta$. Thus, we only need to show that $p+b\mathcal{M}_i \subset b\mathcal{M}_i$. Let $i\in\{1,2,...,5\}$ and $f\in b\mathcal{M}_i$.

For any real number α and r rational, set $A_\alpha=\{x: f(x)+p(x)>\alpha\}$, $B_r=\{x: f(x)>\alpha-r\}$, and $C_r=\{x: p(x)>r\}$. Since $f\in b\mathcal{M}_i$ and $p\in\mathcal{C}$, $B_r\in M_i$ [10], and C_r is open. Hence, $B_r \cap C_r$ is in M_i. Since $A_\alpha=\cup\{B_r \cap C_r$: r is rational$\}$, it follows that $A_\alpha\in M_i$. Hence, $f+p\in b\mathcal{M}_i$. This completes the proof.

B. **Typical Properties in L-families:** We shall now discuss the typical behavior of functions in an L-family. In particular, among other results, we show that a typical function in an L-family has every extended real number as a derived number at every point. To carry out this discussion, some notation is necessary.

Let s and t be real numbers with $t>0$. Let k be a natural number greater than 2, and let (x_0,y_0) be any point in the plane.

The set $K^+(x_0,y_0;s,t)$ (resp. $K^-(x_0,y_0;s,t)$) denotes all points (x,y) in the plane such that $x_0<x<x_0+t$ (resp. $x_0-t<x<x_0$) and $(y-y_0)/(x-x_0)>s$, and the set $K(x_0,y_0;s,t)$ denotes $K^+(x_0,y_0;s,t) \cup K^-(x_0,y_0;s,t)$.

If f is a function defined on I, the set $\mathcal{X}^+(f;s,t)$ denotes all points x in I such that $Gr(f) \cap K^+(x,f(x);s,t)=\emptyset$, and $\mathcal{X}_k^+(f;s,t)$ denotes $\mathcal{X}^+(f;s,t) \cap [1/k,1-1/k]$. The sets $\mathcal{X}^-(f;s,t)$, $\mathcal{X}(f;s,t)$, $\mathcal{X}_k^-(f;s,t)$ and $\mathcal{X}_k(f;s,t)$ are defined in the obvious manner.

If Φ is a subfamily of $b\mathcal{DB}_1$, $A(s,t,k)$ denotes the class of functions f in Φ such that $\mathcal{X}_k(f;s,t)$ is not empty.

Finally, if X is one of the symbols in $\{K^+, K^-, K, \quad \mathcal{X}^+, \mathcal{X}^-, \mathcal{X}\}$ we denote $X(*;k,1/k)$ by $X(*;k)$. If X is one of the symbols in the set $\{\mathcal{X}_k^+, \mathcal{X}_k^-, \mathcal{X}_k\}$ we denote $X(f;k,1/k)$ by $X(f)$, and we denote $A(k,1/k,k)$ by A_k.

To begin with, we prove

Lemma B1. *If Φ is closed in $b\mathcal{DB}_1$, then for all natural numbers $k>2$, A_k is closed in Φ.*

Proof. Fix $k>2$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in A_k that converges to a function $f\in\Phi$. We must show that $f\in A_k$.

First, since $\{f_n\}_{n=1}^{\infty} \subset A_k$, $\mathcal{X}_k(f_n)\neq\emptyset$ for all n. Let the sequence $\{x_n\}_{n=1}^{\infty}$ be such that $x_n\in\mathcal{X}_k(f_n)$ for every n. Clearly, the sequence $\{(x_n,f_n(x_n))\}_{n=1}^{\infty}$ is bounded. Hence, it has a limit point (x,y). We shall show that $y=f(x)$ and $x\in\mathcal{X}_k(f)$.

Suppose that $y<f(x)$. Since $f\in\mathcal{DB}_1$, there exists a point z such that $x<z<x+t$ and $(z,f(z))\in K^+(x,y;k)$. Then, since $f_n \to f$ and $x_n \to x$, it is clear that there exists an $N\geq 1$ such that $|x-x_N|<t$ and the point $(z,f_N(z))$ lies above the line of slope k which contains the point $(x_N,f_N(x_N))$; i.e., $x_N\notin\mathcal{X}_k(f_N)$, which is a contradiction. Hence, $y\geq f(x)$.

Similarly, $y\leq f(x)$. Hence, $y=f(x)$. It is also clear, from the above argument, that $x\in\mathcal{X}_k(f)$. Therefore, $f\in A_k$.

In the next lemma and for the remainder of this paper, $S[x,y;\delta]$ denotes the open square with center (x,y) and side length δ, and whose sides are parallel to the coordinate axes.

Lemma B2. Let f *be in* Φ. *Let* δ *and* ε *be positive real numbers with* $\delta<\varepsilon/4$. *Let* x *in* $(0,1)$ *be such that* $Gr(f) \cap S[x,f(x)+\varepsilon/2;\delta]$ *is empty*.

Then there exists a *and* b *in* I *and a function* u *in* Φ *such that*

(1) $a<b<x$ *and* $\{(a,f(a)),(b,f(b))\} \subset S[x,f(x);\delta]$,

(2) $u\leq f$ *on* (a,b) *and* $u\geq f$ *on* (b,x),

(3) $\{x\colon f(x)\neq u(x)\} \subset (a,b) \cup (b,x)$,

(4) $Gr(u) \cap S[x,f(x)+\varepsilon/2;\delta]\neq\emptyset$,

(5) $Gr(u) \cap S[x,f(x)-\varepsilon/2;\delta]\neq\emptyset$, *and*

(6) $||u-f||<\varepsilon$.

Proof. Since $f\in b\mathcal{DB}_1$, it is clear that we can find points a, b, x_1, x_2 in $\mathrm{dom}S[x,f(x);\delta]$ such that $a<x_1<b<x_2<x$ and the points $(a,f(a))$, $(x_1,f(x_1))$, $(b,f(b))$, and $(x_2,f(x_2))$ are all in $S[x,f(x);\delta]$. Define

$$p(x) = \begin{cases} 0 & \text{if } x\in(a,b)\cup(b,x), \\ f(x)-f(x_1)-\varepsilon/2 & \text{if } x = x_1, \\ f(x)-f(x_2)+\varepsilon/2 & \text{if } x = x_2, \\ \text{linear on } (a,x_1),\ (x_1,b),\ (b,x_2) \text{ and } (x_2,x). \end{cases}$$

Let $u=f+p$. Since Φ is an L-family, $u\in\Phi$. Clearly, u satisfies (1), (2), and (3). Moreover, since $u(x_1)=f(x)-\varepsilon/2$ and $u(x_2)=f(x)+\varepsilon/2$, u satisfies (4) and (5). Finally, since $x_1,x_2\in\mathrm{dom}S[x,f(x);\delta]$ and $\delta<\varepsilon/4$, we have $|f(x)-f(x_i)|<\varepsilon/4$ for $i=1,2$. Hence, $||u-f||<\varepsilon$. This completes the proof.

Theorem B1. The class of functions $f\in\Phi$ having both ∞ and $-\infty$ as derived numbers at each point x in $(0,1)$ is residual in Φ.

Proof. Let A (resp. A') consist of all functions $f\in\Phi$ for which there exists x in $(0,1)$ such that ∞ (resp. $-\infty$) is not a derived number from either side at x. We need to show that $A \cup A'$ is an F_σ of first category in Φ. For this, it is enough to show that A is an F_σ of first category. Clearly, $A=\cup_{k>2}A_k$. Hence, we only need to show that A_k is closed and nowhere dense for every k.

Fix k. By Lemma B1, A_k is closed in Φ. Thus, it suffices to show that A_k is also nowhere dense. To do this, we take $f\in\Phi$ and $\varepsilon>0$, and we find a function $g\in\Phi$ such that $||f-g||<\varepsilon$ and $\mathcal{X}_k(g)=\emptyset$.

First, we prove that there exists a finite set $F=\{y_1,...,y_n\}$ such that, if $x\in\mathcal{X}_k(f)$, there exists a $y_i\in F$ and positive numbers $\varepsilon(y_i)$ and $\delta(y_i)$ such that $\delta(y_i)<\varepsilon(y_i)/4$ and $S[y_i,f(y_i)+\varepsilon(y_i)/2;\delta(y_i)] \subset K^+(z,f(z);k)$ for all points z in $(x-\delta(y_i),x+\delta(y_i)) \cap \mathcal{X}_k(f)$.

To do this, let $x\in\mathcal{X}_k(f)$. As remarked in [6], $\mathcal{X}_k(f)$ is closed and $f|\mathcal{X}_k(f)$ is continuous. Since $f\in\mathcal{DB}_1$, there exists a point $y>x$ and positive numbers $\varepsilon(y)$ and $\delta(y)$ such that $\delta(y)<\varepsilon(y)/4$ and $S[y,f(y)+\varepsilon(y)/2;\delta(y)] \subset K^+(z,f(z);k)$ for all z in $(x-\delta(y),x+\delta(y)) \cap \mathcal{X}_k(f)$. Let $U(x)=(x-\delta(y),x+\delta(y))$. Then, the collection $\{U(x): x\in\mathcal{X}_k(f)\}$ is an open cover of the compact set $\mathcal{X}_k(f)$. Hence, there exist $U(x_1),...,U(x_n)$ which cover $\mathcal{X}_k(f)$. Clearly, the set $F=\{y_1,y_2,..,y_n\}$ is the desired set.

Let $\delta_1=(1/2)\min\{|s-t|: s\neq t \text{ and } s,t\in\{y_1,\ldots,y_n\}\}$, $\delta_2=(1/2)\min\{\delta(y_1),\ldots,\delta(y_n)\}$, and $\delta=(1/2)\min\{1/k,\delta_1,\delta_2\}$. Then, $\operatorname{dom}S[y_i,f(y_i);\delta] \cap \operatorname{dom}S[y_j,f(y_j);\delta]=\emptyset$ if $i\neq j$.

For each $i\in\{1,2,\ldots,n\}$, let u_i be a function in Φ satisfying (1) through (6) of Lemma B2 with $x=y_i$, $\varepsilon=\varepsilon(y_i)$, and δ as defined above. Define

$$g(x) = \begin{cases} u_i(x) & \text{if } y_i-\delta<x<y_i+\delta, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g\in\Phi$ and $||f-g||<\varepsilon$. It remains to show that $\mathcal{X}_k(g)=\emptyset$.

Let $x\in[1/k,1-1/k]$. By (4) and (5) of Lemma B2, $x\in[1/k,1-1/k]\setminus\mathcal{X}_k(g)$ for all $x\in\mathcal{X}_k(f) \cup \{x: f(x)\neq g(x)\}$. Thus, we may assume that $g(x)=f(x)$ and $x\notin\mathcal{X}_k(f)$.

By the definition of $\mathcal{X}_k(f)$, $x\notin\mathcal{X}_k(f)$ implies that there exists a point z in (0,1) such that $(z,f(z))\in K(x,f(x);k)$. If $g(z)=f(z)$, then $x\in[1/k,1-1/k]\setminus\mathcal{X}_k(g)$ and we are done. Hence, we may assume that $g(z)\neq f(z)$. Then there exists an $i\geq 1$ such that $g(z)=u_i(z)$. By Lemma B2, there exist points a and b such that $a<b<x$, and either $a<z<b$ and $u_i(z)<f(z)$ or $b<z<x$ and $u_i(z)>f(z)$. Moreover, either $(z,f(z))\in K^+(x,f(x);k)$ or $(z,f(z))\in K^-(x,f(x);k)$.

Assume that $(z,f(z))\in K^+(x,f(x);k)$. Then, if $b<z<x$, $g(z)=u_i(z)>f(z)$. Hence, $(z,g(z))\in K^+(x,f(x);k)$ and we are done. Thus, we may assume that $a<z<b$. By (4) of Lemma B2, there exists z' in the interval (b,x) such that $u_i(z')>f(z)$. Then, $(z',g(z'))\in K^+(x,f(x);k)$ and $x\in[1/k,1-1/k]\setminus\mathcal{X}_k(g)$.

Similarly, $(z,f(z))\in K^-(x,f(x);k)$ implies that

$x\in[1/k,1-1/k]\setminus \chi_k(g)$. Therefore, $\chi_k(g)=\emptyset$. This completes the proof.

Theorem B2. The class E of functions f in Φ having both ∞ and $-\infty$ as derived numbers at every point x in I is residual in Φ.

Proof. Let E_1 be the class of Theorem B1 which is residual in Φ. The class E_2 of functions $f\in\Phi$ having both ∞ and $-\infty$ as derived numbers at 0 and 1 are residual in Φ, so it follows that $E= E_1 \cap E_2$ is residual in Φ, completing the proof.

To prove Theorems B3 and B4 we need

Lemma B3. Let Φ be an L-family. Let f be in Φ. Let $\delta>0$, and $(x,y)\in(0,1)\times R$ be such that $|y-f(x)|>2\delta$.

If $y-f(x)>2\delta$, then there exists a point a, with $a<x$ and $(a,f(a))\in S[x,f(x);\delta]$, and a function $u\in\Phi$ such that

(1) $\left|\liminf_{t\to x-} f(t)-\inf\{f(t):\ a<t<x\}\right| < \delta/2$,

(2) $u\geq f$ on (a,x) and $u=f$ on $I\setminus(a,x)$,

(3) $Gr(u)\cap S[x,y;\delta]\neq\emptyset$, and

(4) $||u-f||<|y-f(x)|+\delta$.

If $f(x)-y>2\delta$, then there exists a point a in $\mathrm{dom}S[x,f(x);\delta]$, with $a<x$ and a function $v\in\Phi$ satisfying (1) through (4) with u replacing v, "sup" replacing "inf" in (1), and "$\geq$" replacing "$\leq$" in (2).

The proof of this lemma is similar to that of Lemma B2 and will be omitted. In the next lemma, we use the following notation.

Let n be a fixed positive integer and let (x_0, y_0) be any point in the plane, we define

$$R_n^+(x_0,y_0) = \{(x,y): 0 < x-x_0 < \frac{1}{n}, \text{ and } 0 < \frac{y-y_0}{x-x_0} < \frac{1}{n}\},$$

$$R_n^-(x_0,y_0) = \{(x,y): 0 < x_0-x < \frac{1}{n}, \text{ and } -\frac{1}{n} < \frac{y-y_0}{x-x_0} < 0\},$$

$$K_n(x_0,y_0) = \{(x,y): 0 < |x-x_0| < \frac{1}{n}, \text{ and } \left|\frac{y-y_0}{x-x_0}\right| < \frac{1}{n}\}.$$

A real-valued function f is said to have <u>property <n> at a point</u> (x_0, y_0) if $x_0 \in \mathrm{dom} f$ and $\mathrm{Gr}(\bar{f}) \cap K_n(x_0,y_0) \neq \emptyset$. We say f has <u>property <n> on a set</u> E if it has property <n> at every point of E.

<u>Lemma</u> B4. <u>Assume</u> $f \in \Phi$, $\varepsilon > 0$, <u>and</u> k <u>and</u> n <u>are positive integers with</u> $k > 2$. <u>Then there exists a function</u> $g \in \Phi$ <u>and a number</u> $\delta > 0$ <u>such that</u>

(1) $||f-g|| < \varepsilon$, <u>and</u> g <u>has property <n></u> <u>on</u> $\mathrm{Gr}(g|[1/k, 1-1/k])$,

(2) <u>if</u> $h \in \Phi$ <u>and</u> $||g-h|| < \delta$, <u>then</u> h <u>has property <n></u> <u>on</u> $\mathrm{Gr}(h|[1/k, 1-1/k])$.

<u>In particular, the class of functions</u> $f \in \Phi$ <u>having property <n></u> <u>on</u> $\mathrm{Gr}(f|[1/k, 1-1/k])$ <u>is residual in</u> Φ.

<u>Proof</u>. Let $A = \mathrm{cl}\, f|[1/k, 1-1/k]$. Since $f \in \mathcal{DB}_1$, for each $z \in A$ we

can find $z'=(x',y')$ in $(0,1)\times R \cap [R_n^+(z) \cup R_n^-(z)]$ such that

(a) $|y'-f(x')|<\varepsilon/2$.

Then, it is clear that, we can find $\delta(z)$, with $0<\delta(z)<1/n$ and

(b) $w\in S[z;\delta(z)]$ implies $S[z';\delta(z)] \subset R_n^+(w) \cup R_n^-(w)$,

(c) $w\in S[z';\delta(z)]$ implies $S[z;\delta(z)] \subset K_n(w)$.

The collection $\{S[z;\delta(z)/2]: z\in A\}$ is an open cover of the compact set A, so there is a finite subcollection $S[z_1;\delta(z_1)/2], S[z_2;\delta(z_2)/2], \ldots, S[z_m;\delta(z_m)/2]$ which covers A. Then, clearly, we can redefine $z_1', z_2', \ldots, z_m'$ to all have distinct first coordinates and still satisfy (a) through (c) above.

Let $\delta_1=(1/4)\min\{\varepsilon, |y_i'-f(x_i')|, 1/n, \delta(z_i)(1\le i\le m)\}$, $\delta_2=(1/4)\min\{|s-t|: s\ne t,\ s,t\in\{x_1,x_2,\ldots,x_m,x_1',x_2',\ldots,x_m'\}\}$, and $\delta=\min\{\delta_1.\delta_2\}$. Clearly, $\mathrm{dom}S[z_i';\delta] \cap \mathrm{dom}S[z_j';\delta]=\emptyset$ if $i\ne j$.

Let $M_1=\{i: 1\le i\le m,\ y_i'<f(x_i')\}$ and $M_2=\{i: 1\le i\le m,\ y_i'>f(x_i')\}$. For each $i\in M_1$ (resp. $i\in M_2$) let u_i (resp. v_i) be the function of Lemma B3 with $x=x_i'$, $y=y_i'$, and δ as defined above, and define

$$g(x) = \begin{cases} u_i(x) & \text{if } x_i'-\delta<x<x_i'+\delta,\ i\in M_1, \\ v_i(x) & \text{if } x_i'-\delta<x<x_i'+\delta,\ i\in M_2, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, $g\in\Phi$, and by (3) of Lemma B3, $||f-g||<\varepsilon/2+2\delta<\varepsilon$.

We now prove (1) and (2) of this Lemma. For this, we show that if $h\in\Phi$ satisfies $||g-h||<\delta$, then h has property <n> on $Gr(h|[1/k,1-1/k])$.

First, since $Gr(g) \cap S[z_i';\delta]\ne\emptyset$ when $1\le i\le m$, and since

$2\delta<\delta(z_i)$, we have $Gr(h) \cap S[z_i';\delta(z_i)]\neq\emptyset$ when $1\leq i\leq m$. Hence, by (b) and (c) above, h has property <n> on the set $T=Gr(h) \cap \bigcup_{i=1}^{m} \{S[z_i;\delta(z_i)] \cup S[z_i';\delta(z_i)]\}$.

Let $(x,h(x))\in Gr(h|[1/k,1-1/k])\setminus T$. Then, there exists an i such that $g(x)=u_i(x)$ or $g(x)=v_i(x)$. If not, then $g(x)=f(x)$, and since $||g-h||<\delta$, $|f(x)-h(x)|<\delta$, forcing $(x,h(x))$ to be in some $S[z_i;\delta(z_i)]$ contradicting the choice of $(x,h(x))$.

First, assume that $g(x)=u_i(x)$, and let $(y,h(y))\in S[z_i';\delta(z_i)]$. By (3) of Lemma B3, since $(x,h(x))\notin S[z_i';\delta(z_i)]$, it follows that $h(y)>h(x)$. Using Lemma B3 and the fact that $x\notin T$, we can find a point a in $\operatorname{dom}S[z_i,\delta(z_i)]$ such that $a<x<x_i$ and exactly one of the following is true:

(α) $h(x) > \max\{h(a),h(x_i)\}$;

(β) $h(x) < \min\{h(a),h(x_i)\}$.

If (α) is true, then, since $h(y)>h(x)$ and h is Darboux, h crosses the horizontal line $y=h(x)$ in at least two distinct points, one between a and y and the other between y and x_i. Hence, $Gr(h) \cap K_n(x,h(x))\neq\emptyset$, and h has property <n> at $(x,h(x))$. If (β) is true, we use (1) of Lemma B3 and the fact that h is Darboux to conclude that $Gr(h) \cap K_n(x,h(x))\neq\emptyset$.

A similar argument holds if $g(x)=v_i(x)$. Therefore h has property <n> on $Gr(h|[1/k,1-1/k])$. This completes the proof.

<u>Theorem</u> B3. <u>The class Ψ of functions</u> $f\in\Phi$ <u>having zero as a derived number at each point</u> x <u>in</u> I <u>is residual in</u> Φ.

Proof. For integers $n \geq 1$ and $k > 2$, let $E(n,k)$ be the class of functions $f \in \Phi$ having property <n> on $Gr(f|[1/k, 1-1/k])$. By the previous lemma, each $E(n,k)$ is residual in Φ.

Clearly, the class E of functions $f \in \Phi$ having zero as a derived number at the points 0 and 1 is residual in Φ. Since $\Psi = E \cap \bigcap_{n=1}^{\infty} \bigcap_{k=3}^{\infty} E(n,k)$, it follows that Ψ is also residual in Φ.

As a corollary to Theorem B3 we have

Theorem B4. *The class of functions f in Φ having every extended real number as a derived number at every point x in I is residual in Φ.*

Proof. Let Ψ be the class of Theorem B3. For each real number r, let L_r be the function defined by $L_r(x) = rx$ for all points x in I. Put $E_r = \{f + L_r : f \in \Psi\}$. Each E_r is residual in Φ. This follows from the easily proven fact that $N + L_r = \{f + L_r : f \in N\}$ is nowhere dense in Φ, if N is nowhere dense in Φ. Clearly, the family of functions $\cap \{E_r : r \text{ is rational}\}$ is the desired family.

Definition B1. A real-valued function f defined on I is said to be *nowhere monotonic* on I if it is not monotonically increasing or decreasing on any subinterval J of I.

Theorem B5. *The class of functions f in Φ such that $f(x) + rx$ is nowhere monotonic for every real number r is residual in Φ.*

Proof. This class is a superset of the class, Ψ, in Theorem B4. Let $f\in\Psi$. Then $f+L_r$ has all real numbers as derived numbers at every point x in I.

Suppose that $f+L_r$ is monotonic on some interval J. Say it is increasing on J. Then f has no negative derived numbers on J, a contradiction, proving Theorem B5.

We close this section with a discussion of the bilateral behavior of derived numbers of a function f in some residual subset of an L-family Φ. For $\Phi \subset b\mathcal{DB}_1$, we denote by Φ_∞ the class of functions $f\in\Phi$ having ∞ and $-\infty$ as derived numbers at each point $x\in I$. We begin with

Lemma B5. For any positive integer k greater than 2, the class F_k of functions $f\in\Phi_\infty$ such that $\chi^+(f;k) \cap C(f)$ is closed and nowhere dense in $C(f)$ is residual in Φ_∞.

Proof. Let $E(f;k)=\chi^+(f;k) \cap C(f)$. First, we show that $E(f;k)$ is closed in $C(f)$ whenever $f\in\Phi_\infty$.

Let $x\in C(f)\setminus E(f;k)$. Then $K^+(x,f(x);k)$ contains a point $(t,f(t))$. Clearly, there exists a $\delta>0$ such that $(u,v)\in S[x,f(x);\delta]$ implies $(t,f(t))\in K^+(u,v;k)$. Since f is continuous at x, we can find an open interval J containing x such that $f(J) \subset S[x,f(x);\delta]$. Since $J \cap C(f) \subset C(f)\setminus E(f;k)$, it follows that $E(f;k)$ is closed in $C(f)$.

Now, for an interval I with rational endpoints, define $A(I;k)=\{f\in\Phi_\infty\colon I \cap C(f) \subset E(f;k)\}$. Since each $E(f;k)$ is closed in $C(f)$, it follows that $F_k=\Phi_\infty \setminus \bigcup_I A(I;k)$, where the union is taken over all intervals I with rational endpoints. To complete the proof, it suffices to show that each $A(I;k)$ is closed and nowhere dense in Φ_∞.

Let us show that $A(I;k)$ is closed in Φ_∞. To this end, let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $A(I;k)$ converging to a function $f\in\Phi_\infty$. We must show that $f\in A(I;k)$.

Suppose that $f\in\Phi_\infty\setminus A(I;k)$. Then, there exists a point $x\in[I \cap C(f)]\setminus E(f;k)$ such that $Gr(f) \cap K^+(x,f(x);k)\neq\emptyset$. Let $(t,f(t))\in K^+(x,f(x);k)$. Then there exists a number $\delta>0$ such that $(x-\delta,x+\delta) \subset I$, and if $(u,v)\in S[x,f(x);\delta]$ then $S[t,f(t);\delta] \subset K^+(u,v;k)$. Since $f_n \to f$ as $n \to \infty$, there exists an $N\geq 1$ such that $(t,f_N(t))\in S[t,f(t);\delta]$. Moreover, since $\bigcap_{n=1}^\infty C(f_n)$ is residual in I, there exists $y\in\bigcup_{n=1}^\infty C(f_n)$ such that $(y,f_N(y))\in S[x,f(x);\delta]$. But this implies that $f_N\notin A(I;k)$, which is a contradiction. Therefore $A(I;k)$ is closed in Φ_∞.

Now we show that if $f\in\Phi_\infty$ and $\varepsilon>0$, then there exists a function $u\in\Phi_\infty\setminus A(I;k)$ such that $||u-f||<\varepsilon$. That is, $A(I;k)$ is nowhere dense in Φ_∞. Obviously, we may assume that $f\in\Phi_\infty\setminus A(I;k)$.

Let $t\in I \cap C(f)$. Then there exists a number η, with $0<\eta<1/k$ and $(t,t+\eta) \subset I$, and a point z in $(t,t+\eta)$ such that $(z,f(z)+\varepsilon/2)\in K^+(t,f(t);k)$.

Choose $\delta>0$ such that $(z-\delta,z+\delta) \subset (t,t+\eta)$ and $\delta<\varepsilon/4$. Since Φ_∞ is an L-family, there exists a function $u\in\Phi_\infty$ satisfying the conclusions of Lemma B3, with $x=z$ and $y=f(z)+\varepsilon/2$. Moreover, since $f(t)=u(t)$ and u and f have the same continuity points, we have that $t\in[I \cap C(u)]\setminus E(f;k)$. Hence $u\in\Phi_\infty\setminus A(I;k)$ and by (3) of Lemma B3, $||u-f||\in\varepsilon$. This completes the proof.

<u>Theorem</u> B6. <u>There exists a residual subset</u> Ψ <u>of</u> Φ <u>such that for every</u> $f\in\Psi$ <u>there exists a residual subset</u> $E(f)$ <u>of</u> I <u>such that every extended real number is a bilateral derived number of</u> f <u>at each</u> $x\in E(f)$.

<u>Proof</u>. For each $f\in\Phi_\infty$, let $E_+(f)$ (resp. $E_-(f)$) denote the set of points $x\in C(f)$ such that $-\infty$ is a derived number from the right (resp. left) at x, but that ∞ is not a derived number from the right (resp. left) at x. We will show that $E_+(f) \cup E_-(f)$ is of first category for every function in some residual subset of Φ_∞.

For each $k>2$, let F_k be the residual subset of Φ_∞ obtained from Lemma B5. Let $F=\bigcap_{k=3}^{\infty}F_k$. Clearly, F is residual in Φ_∞. Moreover, if $f\in F$, then $E_+(f)=\bigcup_{k=3}^{\infty}E(f;k)$ which is of first category in $C(f)$, and hence in I.

Similarly, there exists a residual subset F' of Φ_∞ such that for every $f\in F'$ the set $E_-(f)$ is of first category in I. If we put $\Psi=F \cap F'$, then it follows that Ψ is residual in Φ_∞. By Theorem B1, Φ_∞ is residual in Φ. It follows that Ψ is residual in Φ. Clearly, Ψ is the desired set and the proof is complete.

Two questions arise in connection with Theorem B6.

Question B1. Given an L-family Φ, does there exist a residual subset Ψ of Φ such that if $f \in \Psi$, then every real number is a bilateral derived number of f at every point x in I ?

Question B2. Given an L-family Φ, does there exist a residual subset Ψ of Φ and a residual subset E of I such that if $f \in \Psi$, then every real number is a bilateral derived number of f at every point of E ?

The first question has a negative answer for $\mathcal{C}$, $b\mathcal{D}usc$, $b\mathcal{D}lsc$, $b\Delta$, and $b\mathcal{M}_i$ $(i=1,2,\ldots,5)$. This is a consequence of a theorem of M. Chlebik, [5] Lemma 5, which implies the following

Theorem B7[Chlebik [5]]. *Each of the families* $\mathcal{C}$, $b\mathcal{D}usc$, $b\mathcal{D}lsc$, $b\Delta$, and $b\mathcal{M}_i$ $(i=1,2,\ldots,5)$ *contains a residual subset* Ψ *such that if* $f \in \Psi$, *then* f *attains a relative maximum (and minimum) at exactly one point in each open subinterval of* I *with rational endpoints*.

If Ψ is as in Theorem B7 and $f \in \Psi$, then number 1 is not a derived number at the points where f achieves a maximum. The answer to the second question is still open for many L-families. However, it has a negative answer in the case of $b\mathcal{DB}_1$, as the following theorem shows.

Theorem B8. For every residual subset Ψ of $b\mathcal{DB}_1$ and every residual subset E of I there exists a function $f\in\Psi$ and a point $x\in E$ such that 1 is not a bilateral derived number of f at x.

Proof. Let Ψ be a residual subset of $b\mathcal{DB}_1$ and let E be a residual subset of I. Let F be a bilaterally c-dense-in-itself F_σ subset of E. Then, there exists a function $f\in b\mathcal{DB}_1$ such that $0<f(x)\leq 1$ for $x\in F$, $||f||=1$, and $f(x)=0$ if $x\in I\setminus F$ [1].

By Theorem B7, there exists a function $g\in\Psi$ which attains a relative maximum at exactly one point in each open subinterval of I with rational endpoints, and such that $||f-g||<1/4$. Clearly, g attains its maximum at a point $x\in E$. Therefore the number 1 is not a derived number at x. This completes the proof.

C. **Intersections with Lines:** In this section we consider the size and structure of the set consisting of the intersection of a line, with a given slope, with the graph of a function f. We begin with two definitions, the first of which is due to Bruckner and Garg [3].

Definition C1. A nowhere monotone function is said to be of the second species if $f(x)+rx$ remains nowhere monotone for every real number r.

Definition C2. A subset B of R is called a boundary set if Int $B=\emptyset$.

Theorem C1[Bruckner-Garg [3]]. *If a function* f *in* $b\mathcal{DB}_1$ *is of the second species, then for every countable set* E *of* R *there exists a residual set* H *in* R *such that* $\{x\colon f(x)=rx+s\}$ *is a dense-in-itself boundary set whenever* r *is in* E *and* s *is in* H.

Theorem C2. *There exists a residual set* Ψ *in* Φ *such that for each* f *in* Ψ *there exists a residual set* $H(f)$ *in* R *such that* $\{x\colon f(x)=rx+s\}$ *is a dense-in-itself boundary set whenever* r *is rational and* s *is in* $H(f)$.

Proof. By Theorem B5, the set Ψ^* of functions $f\in\Phi$ of the second species is residual in Φ. Now apply the previous theorem with E the rational numbers.

Theorem C3. *Let* h *be an arbitrary real-valued, continuous function defined on* I. Suppose that $\Phi+h=\Phi$. *Then there exists a residual subset* $\Psi(h)$ *in* Φ *such that for every* f *in* $\Psi(h)$ *there exists a residual set* $H(f)$ *in* R *such that* $\{x\colon f(x)=h(x)+s\}$ *is a dense-in-itself boundary set whenever* s *is in* $H(f)$.

Proof. It is clear, from the hypothesis, that the mapping $\varphi\colon \Phi \to \Phi$ defined by $\varphi(f)=f-h$ is a homeomorphism of Φ onto Φ. By Theorem C2, there exists a residual set $\Psi^* \subset \Phi=\varphi(\Phi)$ such that for every $g\in\Psi^*$ there exists a residual set $H^*(g) \subset R$ such that $\{x\colon g(x)=s\}$ is a dense-in-itself boundary set whenever $s\in H^*(g)$.

Since, $\Psi(h)=\varphi^{-1}(\Psi^*)$ is residual in Φ, it follows that for each $f\in\Psi(h)$ there is a residual set $H(f)=H^*(f-h) \subset R$ such that

$\{x: (f-h)(x)=s\}=\{x: f(x)=h(x)+s\}$ is a dense-in-itself boundary set whenever $s\in H(f)$. This proves the theorem.

Corollary C1. Let $\mathcal{H}$ be a countable family of real-valued, continuous functions defined on I. Suppose that $\Phi+h=\Phi$ for every h in $\mathcal{H}$, then there exists a residual set $\Psi(\mathcal{H})$ in Φ such that for every f in $\Psi(\mathcal{H})$ there exists a residual set H(f) in R such that $\{x: f(x)=h(x)+s\}$ is a dense-in-itself boundary set whenever $h\in\mathcal{H}$ and $s\in H(f)$.

Proof. For each $h\in\mathcal{H}$ there exists, by Theorem C3, a residual set $\Psi(h) \subset \Phi$ such that if $f\in\Psi(h)$, there exists a residual set $H(f,h) \subset R$ satisfying the conclusion of Theorem C3. Let $\Psi(\mathcal{H})=\bigcap_{h\in\mathcal{H}}\Psi(h)$, which is residual in Φ. Finally, if $f\in\Psi(\mathcal{H})$ we only need to take $H(f)=\bigcap_{h\in\mathcal{H}}H(f,h)$.

Corollary C2. Each of the families $\mathcal{C}$, b$\mathcal{D}$usc, b$\mathcal{D}$lsc, bΔ, and b$\mathcal{M}_i$ (i=1,2,..5) satisfies the hypothesis of Theorem C2 and its corollary.

In Theorem 3.2 of [2], Bruckner shows that there exists a residual class N of continuous functions such that for each function f in N there exists a countable dense set $S_f \subset R$ such that the set E_α, defined by, $E_\alpha=\{x: f(x)=\alpha\}$, is a perfect set when $\alpha\in R\setminus S_f$ and is a nonempty perfect set union an isolated point when $\alpha\in S_f$.

We will show that for certain subfamilies Φ of b$\mathcal{DB}_1$ there exists a residual set N of Φ such that for each f in N there exists a countable dense set $S_f \subset R$ such that E_α is a dense-in-itself boundary G_δ set when $\alpha \in R \setminus S_f$ and is a nonempty dense-in-itself boundary G_δ set union an isolated point when $\alpha \in S_f$.

This is an analogue to Bruckner's result since a dense-in-itself boundary G_δ set is homeomorphic to the bilateral limit points of the Cantor set.

Many of the theorems and lemmas appearing below have proofs similar to those found in [3]. We begin with the following

<u>Definition</u> C3. A function f in $b\mathcal{DB}_1$ will be called of <u>oscillatory type</u> if every extended real number is a derived number of f at every point x in I.

<u>Remark</u> C1. As a consequence of Theorem B4, the functions of oscillatory type form a residual subset of any L-family Φ.

<u>Lemma</u> C1. <u>Let</u> Φ <u>be an L-family and let</u> A <u>consist of those functions</u> f <u>in</u> Φ <u>for which no set of the form</u> $\{x: f(x)=\alpha\}$ <u>contains more than one point at which the function achieves a relative extremum</u>. <u>Then</u> A <u>is a residual</u> G_δ <u>in</u> Φ.

<u>Proof</u>. For two disjoint intervals I and J with rational endpoints, let A(I,J) denote the set of functions $f \in \Phi$ for which neither the supremum nor the infimum of f on I is equal to either the supremum or the infimum of f on J. We wish to show

that $A=\cap A(I,J)$ is a dense subset of Φ of type G_δ. For this purpose, it suffices to show that $A(I,J)$ is dense and open in Φ for every pair (I,J).

Suppose that I and J are disjoint closed intervals. Let E_1 denote the class of functions $f\in\Phi$ such that

$$(*) \qquad \sup\{f(x): x\in I\}\neq\sup\{f(x): x\in J\}.$$

Choose $f\in E_1$. Write $\alpha=\sup\{f(x): x\in I\}$ and $\beta=\sup\{f(x):x\in J\}$, and set $\varepsilon=|\alpha-\beta|$. It is clear that if $g\in\Phi$ and $||f-g||<\varepsilon/2$ then $g\in E_1$. Hence, E_1 is open. To see that E_1 is also dense, let $\mathcal{G}$ be an open subset of Φ, then $\mathcal{G}$ contains an open set $\mathcal{G}_h=\{g: ||g-h||<\varepsilon\}$ for some $h\in\mathcal{G}$ and some $\varepsilon>0$. We must show that $E_1 \cap \mathcal{G}_h\neq\emptyset$. To do this, assume that $\sup\{h(x): x\in I\}\geq \sup\{h(x): x\in J\}$. By Lemma B3, there exists a function $g\in\Phi$ such that $||h-g||<\varepsilon$, $\{x: g(x)\neq h(x)\} \subset I$, and $\sup\{g(x): x\in I\}> \sup\{h(x): x\in I\}$. Hence, $g\in E_1$, and since $||g-h||<\varepsilon$, we see $g\in\mathcal{G}_h \subset \mathcal{G}$. Thus, $E_1 \cap \mathcal{G}\neq\emptyset$ and E_1 is dense in Φ.

Now, replacing "sup" by "inf" in one or both sides of the inequality in (*) above, we arrive at the sets E_2, E_3, and E_4 which are also dense and open in Φ. It is clear that $A(I,J)= \bigcap_{i=1}^{4}E_i$ and that $A(I,J)$ is therefore dense in and open in Φ.

<u>Definition</u> C4. A subfamily Φ of $b\mathcal{DB}_1$ is called an <u>L^*-family</u> if it is an L-family and there exists a residual set Ψ of Φ such that each f in Ψ attains a relative maximum (and minimum) at exactly one point in each open subinterval of I.

Lemma C2. The families $\mathscr{C}$, b$\mathscr{D}$usc, b$\mathscr{D}$lsc, bΔ, and b$\mathscr{M}_i$ $(i=1,2,\ldots,5)$ are all L^*-families.

The proof of Lemma C2 is a direct consequence of Theorem B7.

In the next theorem we will use the following notation. Let Φ be an L^*-family and let $f\in\Phi$. We set $M_f=\sup\{f(x):\ x\in I\}$ and $m_f=\inf\{f(x):\ x\in I\}$.

Theorem C4. Let Φ be an L^*-family and let N be the class of functions f in Φ to each of which corresponds a dense denumerable subset S_f of the interval (m_f, M_f) such that $E_\alpha=\{x:\ f(x)=\alpha\}$ is

(1) a dense-in-itself boundary G_δ set when $\alpha\in S_f\setminus\{m_f, M_f\}$,

(2) a single point when $\alpha=m_f$ or M_f,

(3) of the form $C_\alpha \cup \{x_\alpha\}$ where C_α is a nonempty dense-in-itself boundary G_δ set and x_α is an isolated point of E_α.

Then N is residual in Φ.

Proof. Let B be a residual set in Φ such that each f in B attains a relative maximum (and minimum) at exactly one point in each open subinterval of I. Let Ψ be the intersection of B with the residual subset in Remark C1. Then Ψ is residual in Φ and each set E_α of a function f in Ψ is a boundary G_δ set. We will show that $\Psi \subset N$ from which it will follow that N is residual in Φ.

Since f is of oscillatory type, a point will be isolated in some E_α if and only if f achieves a strict extremum at x. It follows, from Lemma C1, that each point of extremum is a strict point of extremum. Since a function $f \in \Psi$ attains a point of extremum in each open subinterval of I, it follows that the set, D, of points of extremum of a function $f \in \Psi$ is dense in I. Moreover, since each point of extremum is a strict point of extremum, D is denumerable.

We now show that $f(D)$ is a denumerable set dense in (m_f, M_f). Clearly, $f(D)$ is denumerable. If $f(D)$ is not dense, then there exists an interval $(c,d) \subset (m_f, M_f)$ for which $f(D) \cap (c,d) = \emptyset$. Pick $\delta > 0$ so that $\delta < (d-c)/2$ and let $E = \mathrm{cl}\, f^{-1}(c+\delta, d-\delta)$. Clearly, E is a nonempty perfect set.

Choose x to be a point in E at which $f|E$ is continuous. This is possible since $f \in \mathcal{DB}_1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $f^{-1}(c+\delta, d-\delta)$ approaching x. Since Φ is an L^*-family, the function f achieves a maximum and a minimum on each interval of the form (x_n, x_{n+1}). Since $f(D) \cap (c,d) = \emptyset$, it follows that the image of the extrema points on (x_n, x_{n+1}) lie outside (c,d), and since $x_n \to x$ as $n \to \infty$ and f is Darboux, it follows that the interval $[c,d]$ is contained in a cluster set of f at x. It follows that (c,d) is contained in the cluster set of $f|E$ at x. This contradicts the continuity of $f|E$ at x.

Now, let $S_f = f(D) \setminus \{m_f, M_f\}$. Then, for any real number α, if $\alpha \notin S_f \cup \{m_f, M_f\}$ then, since f is of oscillatory type, E_α contains no isolated points. Hence, E_α is a dense-in-itself boundary G_δ set. If $\alpha=m_f$ or M_f, then E_α is a single point since the maximum and minimum of f over the interval I are unique. Finally, if $\alpha \in S_f$ the E_α contains exactly one point of extrema, x_α. The point x_α is isolated, and since $m_f<\alpha<M_f$ and since f is Darboux there are other points of E_α. Since none of these points are isolated, it follows that $E_\alpha \setminus \{x_\alpha\}$ is dense-in-itself. This completes the proof.

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