## Monotonicity and the Approximate Symmetric Derivative

Lee Larson, Department of Mathematics, University of Louisville, Louisville, KY 40292

In the following, the symmetric derivative of a real-valued function is written as

$$f^{s}(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

The approximate symmetric derivative, $f^{as}(x)$, is defined in the natural way by replacing the ordinary limit with the approximate limit. The relationship between monotonicity and the ordinary symmetric derivative is quite well understood. This relationship is fairly succinctly summed up by the following theorem.

**Theorem 1.** ([L1]) If $f:\mathbf{R}\to\mathbf{R}$ is such that $f^{s} \geq 0$ everywhere and $C(f)$ is the set on which $f$ is continuous, then $f|_{C(f)}$ is nondecreasing.

Because every symmetrically differentiable function is differentiable almost everywhere ([U], [K]), Theorem 1 shows that the monotonicity behavior of the symmetric derivative is "almost" that of the ordinary derivative.

In the case of the approximate symmetric derivative, the monotonicity question is still largely open. The most reasonable conjecture seems to be that if $f:\mathbf{R}\to\mathbf{R}$ is measurable and $f^{as} \geq 0$ everywhere, then $f$ is monotone when restricted to

$$A(f)=\{x: f \text{ is approximately continuous at } x\}.$$

There is some circumstantial evidence to support this conjecture.

**Theorem 2.** ([M]) If $f$ is a measurable function such that $f^{as} \geq 0$ everywhere, then given an interval $I$, there exists a subinterval $J$ of $I$ such that $f|_{A(f)}$ is nondecreasing on $J$.

The only other successful attack on the monotonicity question with the approximate symmetric derivative is the following theorem.

**Theorem 3.** ([P]) Let $f:\mathbf{R}\to\mathbf{R}$ be a Darboux-Baire 1 function and for each $\alpha\in\mathbf{R}$ define $\Lambda_\alpha(f) = \{x : f(x)=\alpha\}$ and

$$K=\{\alpha : \Lambda_\alpha(f) \text{ contains at most countably many approximate maxima of } f\}.$$

If $f^{as} \geq 0$ everywhere and $K$ is dense in $\mathbf{R}$, then $f$ is nondecreasing.

Theorem 3 is a consequence of a new theorem which will appear in [L2]. (Theorem 3 is actually stated in a somewhat stronger form in [P].) A weaker version of the new theorem is presented below. First, here is some notation.

Let f be a real-valued function and $\alpha \in \mathbf{R}$. Define

$$A_\alpha(f) = \{x : f(x) > \alpha\},\ B_\alpha(f) = \{x : f(x) < \alpha\} \text{ and } C_\alpha(f) = \overline{A_\alpha(f)} \cap \overline{B_\alpha(f)}.$$

If A is a measurable set, then let D(A) be the set of upper density points of A. Denote by L(A) the set of points at which the lower density of A is greater than 1/2. The derived set of A is written as A'.

**Theorem 4.** If f is a Darboux-Baire 1 function such that $f^{as} \geq 0$ everywhere and there is a dense set D such that for each $\alpha \in D$ either $C_\alpha(f) = \emptyset$, or $L(A_\alpha(f)) \cup L(B_\alpha(f))$ is a second category subset of $C_\alpha(f)$, then f is nondecreasing.

We present here a proof for the simpler case when f is continuous, because this proof contains the essential ideas of the proof which which appears in [L2], but also has fewer complications than the general case.

*Proof.* Without loss of generality, we may assume that $f^{as} > 0$ everywhere, for otherwise, we just consider the function $g(x) = f(x) + \varepsilon x$ for arbitrarily small $\varepsilon > 0$. Suppose there are real numbers x and y with $x < y$ such that $f(x) > f(y)$. Choose $\alpha \in (f(y), f(x))$. Since f is a Darboux function, it follows that $C_\alpha(f) \neq \emptyset$, so at least one of $L(A_\alpha(f))$ or $L(B_\alpha(f))$ is second category in $C_\alpha(f)$.

We see from its definition that $C_\alpha(f)$ is closed. Because f is continuous, $C_\alpha(f)$ is a subset of $A_\alpha(f)$ so $|C_\alpha(f)| = 0$. Therefore, $C_\alpha(f)$ is nowhere dense. Let P be the derived set of $C_\alpha(f)$. Then P is a nowhere dense perfect set with zero measure.

Suppose that (a, b) is a component of $C_\alpha(f)^c$. Then, one of the following statements must be true:

$$(1)\ (a, b) \cap A_\alpha(f) = \emptyset;\ \text{or},\ \ (2)\ (a, b) \cap B_\alpha(f) = \emptyset.$$

Suppose that (1) is true and that $b \in C_\alpha(f)$. Since $f^{as}(b) > 0$, it follows that $A_\alpha(f)$ has density equal to 1 at b. On the other hand, since $b \in C_\alpha(f)$, it follows that b is a right limit point of $B_\alpha(f)$. From this we are forced to conclude that $b \in P$. Similarly, if (2) is true and $a \in C_\alpha(f)$, then $B_\alpha(f)$ has density equal to 1 at $a$ and $a \in P$. Since one of these two must be true for any component $(a, b)$ of $C_\alpha(f)^c$, we see that both $D(A_\alpha(f))$ and $D(B_\alpha(f))$ must be dense in P. But, both of these sets are then dense $G_\delta$ subsets of the Baire space, P, and as such they are residual in P.

An argument similar to that given above shows that if $t$ is an isolated point of $C_\alpha(f)$, then there must be a $\delta>0$ such that $(t-\delta, t) \subset B_\alpha(f)$ and $(t, t+\delta) \subset A_\alpha(f)$, so no isolated point of $C_\alpha(f)$ can be in $L(A_\alpha(f)) \cup L(B_\alpha(f))$. This shows that at least one of $L(A_\alpha(f))$ or $L(B_\alpha(f))$ is a second category subset of P. This is a contradiction because $L(A_\alpha(f)) \cap D(B_\alpha(f)) = \emptyset$ and $L(B_\alpha(f)) \cap D(A_\alpha(f)) = \emptyset$. Therefore, the theorem follows.

## REFERENCES

[K] A. Khintchine, Recherches sur la structure des fonctions measurables, *Fund. Math.*, 9(1927), 212-279.

[L1] L. Larson, The Symmetric Derivative, *Trans. Amer. Math. Soc.*, 277(2), 1983, 589-599.

[L2] L. Larson, A monotonicity theorem for the approximate symmetric derivative, (in preparation).

[M] J. Matousek, The Approximate Symmetric Derivative and Monotonicity, *Comm. Math. Univ. Carol.*, 27,1(1986), 27-30.

[P] H. H. Pu and H. W. Pu, On approximate Schwarz derivatives, *Rev. Roum. Math. Pures Appl.*, 25(1980)2, 257-264. MR 81g #26003.

[U] J. Uher, Symmetrically differentiable functions are differentiable almost everywhere, *Real Anal. Exch.*, 8(1), 1982-83, 253-260.