Integral inequalities for solutions of some partial differential equations *)

by

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There are many papers and monographs devoted to the maximum principle for solutions of partial differential equations (pdes) or systems of pdes (see for example [PW], [Sp]). If a system of pdes is linear, then, in some cases, a nice geometric interpretation of the maximum principle can be given. The purpose of the talk is to give some properties, from the maximum principle point of view, of solutions of a linear pde in two or three real variables.

<u>Formulation of problems</u>. Let M be a smooth, paracompact, separable manifold, $\dim_{\mathbb{R}} M = m$, with a distinguished ℓ - dimensional complex subbundle H of $\mathbb{C}T(M)$. In the following we assume that the bundle H is formally integrable, i.e. is closed with respect to the Poisson bracket.

Denote by S_H the space of smooth functions u on M annihilated by all sections of the bundle, which means, in local coordinates $(x_1,\ldots,x_m)$ on U, $U \subset M$, that u satisfies the system

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$$L_\alpha u = a_{\alpha 1}(x)\frac{\partial u}{\partial x_1}(x) + \dots + a_{\alpha m}(x)\frac{\partial u}{\partial x_m}(x) = 0, \ \alpha = 1,\dots,\ell, \ x \in U,$$

where L_α, $\alpha = 1,\dots,\ell$, are independent sections of H over U, and $a_{\alpha\beta} = a_{\alpha\beta}(x)$ are complex - valued smooth functions on U.

One of the main frequently required properties of solutions to such a system is the maximum principle. Of course the maximum principle is not true for functions from S_H, without additional assumptions on H. Some cases are well - known. For instance if H is a complex structure on M, or if (M,H) is an embedded into $\mathbb{C}^n$ 1 - concave CR manifold, but the general situation seems to be difficult.

Now we formulate the following problems.

Problem 1. What conditions on H ensure that any function $u \in S_H$ satisfies the maximum principle?

Problem 2. Under what conditions on H does there exist a constant $C = C_H$ such that for any $u \in S_H \cap L^1(M)$

$$|u(p)| \leqslant C \int_M |u| \, d\sigma, \tag{1.1}$$

where $d\sigma$ is a volume element on M and $L^1(M)$ is the space of integrable functions on M.

The property (1.1) is stronger than the maximum principle for functions from $S_H \cap L^1(M)$. If the volume of M is finite, then (1.1) gives also

$$|u(p)| \leqslant \tilde{C} \left(\int_M |u|^2 \, dV \right)^{1/2} \quad \text{for } S_H \cap L^2(M), \tag{1.2}$$

the inequality which is needed in the Bergman - Szegö type theory for CR functions, see [D_2]

In general it is too much to require the maximum principle or the property given in Problem 2. Very simple example can be given (e.g. a one dimensional bundle H over $U \subset \mathbb{R}^3$) that these properties fail. However for some type of linear pdes in three variables a nice necessary and sufficient condition for the maximum principle for solutions was given by Hill [H].

Sometimes it is more natural to ask the following:

<u>Problem 3</u>. Under what conditions on H and M does there exist for an arbitrary compact submanifold $N \subset M$ of real dimension $m - 2\ell$ a constant $C = C_N$ such that

$$(1.3) \qquad \left| \int_N u \, d\omega \right| \leqslant C \int_M |u| \, d\sigma , \qquad u \in S_H \cap L^1(M),$$

where $d\omega$ is a volume element on N generated by $d\sigma$.

The last property is a type of maximum principle for the function $\int_N u \, d\omega$ where N varies in M. This property does not seem to be trivial.

<u>The case of a linear pde in two variables</u>.

(1) Let $L = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2}$, $x \in U \subset \mathbb{R}^2$, be a vector field on a domain U, where a_1, a_2 are smooth complex - valued functions. If L, $\bar{L}$ are linearly independent at each $p \in U$, then the maximum principle and inequality (1.1) hold for solutions of $Lu = 0$.

(2) Take $M_k = \frac{\partial}{\partial t} + it^k \frac{\partial}{\partial x}$, $(t,x) \in R^2$, $k = 0,1,2,\dots$.

For these operators the maximum principle fails for odd integers k and holds for even integers k.

<u>The case of a linear pde in three variables</u>. Let M be a domain in $\mathbb{R}^3$, N is a Jordan curve lying in M and the bundle H is generated by a complex vector field L of the form

$$L = \lambda_1(x)\frac{\partial}{\partial x_1} + \lambda_2(x)\frac{\partial}{\partial x_2} + \lambda_3(x)\frac{\partial}{\partial x_3}, \qquad x \in M \subset \mathbb{R}^3,$$

where λ_α are smooth complex - valued functions. The maximum principle for solutions of $Lu = 0$ in general fails. Under some additional assumptions on L the inequality (1.3) holds for some curves N, i.e. the average of solutions of $Lu = 0$ over N can be estimated by a volume integral over a neighborhood of N (for details see [D_1]).

[D_1] R. Dwilewicz, Some remarks on maximum type principles for solutions of linear pdes in two and three variables. Preprint #1403, Centre de recherches mathématiques, Université de Montréal, (1986).

[D_2] R. Dwilewicz, Bergman - Szegö type theory for CR structures. Preprint of the Centre de recherches mathématiques, Université de Montréal (August 1986).

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