Real Analysis Exchange Vol. 11 (1985-86)

Vasile Ene, Institute of Mathematics, str. Academiei 14, 70109, Bucharest, Romania

ON SOME CLASSES OF CONTINUOUS FUNCTIONS

In [3] J. Foran introduced conditions $A(N)$ and $B(N)$, and in [1] we defined condition $E(N)$ for a function on a set E for some positive integer N.

In the present paper we construct a continuous function G_N which satisfies $E(N+1)$ on a perfect set and which is $E(N)$ on no portion of this set. Given a natural number N, let $\mathcal{A}(N)$ (respectively $\mathcal{B}(N)$, $\mathcal{E}(N)$) be the class of all continuous functions F defined on a closed interval I for which there exist a sequence of sets $\{E_n\}$ and natural numbers $\{N_n\}$ such that $\sup(N_n) = N$, $I = \cup E_n$ and F is $A(N_n)$ (respectively $B(N_n)$, $E(N_n)$) on E_n. (If we drop the condition $\sup(N_n) < \infty$ we obtain the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, which were defined in the same articles.) Let us recall that $\mathcal{A}(1) =$ ACG. By the Baire Category Theorem ([5], p. 54), our result means that the class $\mathcal{E}(N)$ is strictly contained in $\mathcal{E}(N+1)$. (We showed in [1] that $\mathcal{A}(N)$ is strictly contained in $\mathcal{A}(N+1)$.) Moreover the continuous function G_N, constructed for this purpose, has also the following properties: $G_N \in \mathcal{A}(N^2+2N+1)$ and $G_N \notin \mathcal{B}(N^2+2N)$.

We construct also a continuous function F which satisfies Foran's condition $\bar{N}$ and $F \notin \mathcal{E}$. (We showed in [2] that $\mathcal{E}$ is strictly contained in $\bar{N}$, but here we have an explicit example.)

Definition 1. Given a positive integer N and a set E, a function F is said to be $B(N)$ on E if there is a number $M < \infty$ such that for any sequence $I_1,\dots,I_k,\dots$ of nonoverlapping intervals with $I_k \cap E \neq \emptyset$, there exist intervals J_{kn}, $n = 1,\dots,N$, such that

$$B(F;E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < M.$$

(Here $B(F;X)$ is the graph of F on the set X.)

Definition 2. Given a positive integer N and a set E, a function F is said to be $A(N)$ on E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $I_1, \ldots, I_k, \ldots$ are nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum_k |I_k| < \delta$, then there exist intervals J_{kn}, $n = 1,2,\ldots,N$, such that

$$B(F;E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon.$$

Definition 3. Given a positive integer N and a set E, a function F is said to be $E(N)$ on E if for every subset S of E, $|S| = 0$, and for each $\varepsilon > 0$ there exist rectangles $D_{kn} = I_k \times J_{kn}$, $n = 1,2,\ldots,N$, where $\{I_k\}$ is a sequence of nonoverlapping intervals, $I_k \cap S \neq \emptyset$ such that

$$B(F;S) \subset \bigcup_k \bigcup_{n=1}^{N} D_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^{N} (\text{diam } D_{kn}) < \varepsilon.$$

Definition 4. [4] $\bar{N}$ denotes the class of real valued functions whose graph on any set of Lebesgue measure 0 is of linear measure 0.

We need also the following preliminary facts:

Let N be a positive integer and let us define on $[0,1]$ the following perfect set:

$C_N = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{c_i}{(2N+1)^i}$, $c_i \in \{0,2,\ldots,2N\}$, for each $i = 1,2,\ldots\}$. Each $x \in C_N$ is uniquely represented by $\sum_{i=1}^{\infty} \frac{c_i(x)}{(2N+1)^i}$.

Clearly C_1 is identical to the Cantor ternary set C. Let

$\varphi_N : [0,1] \to [0,1]$ be defined as follows: For each $x \in C_N$, $\varphi_N(x) = (1/2) \sum_{i=1}^{\infty} \frac{c_i(x)}{(N+1)^i}$. Then φ_N is continuous on C_N. Extending φ_N by linearity on each interval contiguous to C_N, we have φ_N defined and continuous on $[0,1]$. Clearly φ_1 is identical to the Cantor ternary function φ. But φ_N is also increasing on $[0,1]$ and constant on each interval contiguous to C_N. Indeed, let $x,y \in C_N$, $x < y$. Let n be the first positive integer such that $c_n(x) + 2 \leq c_n(y)$. Then $c_i(x) = c_i(y)$, $i = 1,2,\ldots,n-1$. We have $\varphi_N(y) - \varphi_N(x) \geq (1/2) \cdot \left(\frac{2}{(N+1)^n} + \sum_{i=n+1}^{\infty} \frac{c_i(y)-c_i(x)}{(N+1)^i} \right) \geq 0$. For each natural number k let $R_k = \sum_{i=k}^{\infty} \frac{2N}{(2N+1)^i}$, and let $J = (a,b)$ be an interval contiguous to C_N.

Then there exist $c_i \in \{0,2,\ldots,2N\}$, $i = 1,2,\ldots,m$ $c_m \geq 2$, such that

$$b = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} \quad \text{and} \quad a = \sum_{i=1}^{m-1} \frac{c_i}{(2N+1)^m} + R_{m+1}.$$ Hence $\varphi_N(a) = \varphi_N(b)$.

Theorem 1. *Given a positive integer* $N \neq 0$, *there exists a continuous function* G_N *on* $[0,1]$ *which is*: a) $E(N+1)$ *on* C_N; b) $E(N)$ *on no portion of* C_N; d) $A(N^2+2N+1)$ *on* C_N; d) $B(N^2+2N)$ *on no portion of* C_N.

Proof. Let $\{j_n\}$ be a strictly increasing sequence of positive integers, $j_0 = 0$. Let $\{a_n\}$ be a strictly decreasing sequence of positive real numbers, $a_0 = 1$, $\lim a_n = 0$. Let $G_N : C_N \to R$, $G_N(x) = (1/2N) \cdot \sum_{k=0}^{\infty} c_{j_{k+2}}(x) \cdot (a_k - a_{k+1})$. Then G_N is continuous on C_N. Extending G_N linearly on each interval contiguous to C_N, we get G_N defined and continuous on $[0,1]$.

a) We show that, if $a_k \leq 1/(2N+1)^{j_k}$, then G_N is $E(N+1)$ on C_N. Let p be a positive integer, $p \neq 0$ and

$$\mathcal{a}_p = \left\{ x \in C_N : x = \sum_{i=1}^{j_p} \frac{c_i(x)}{(2N+1)^i} \right\}.$$ For each $x \in C_N$, let $I_{x,p} = [x, x+R_{j_p+1}]$.

Then $\mathcal{a}_p$ has $(N+1)^{j_p}$ elements and $|I_{x,p}| = 1/(2N+1)^{j_p} = R_{j_p+1}$. Let

$$J^j_{x,p} = \left[G_N\left(x + \frac{2j}{(2N+1)^{j_p+1}}\right), G_n\left(x + \frac{2j}{(2N+1)^{j_p+1}} + R_{j_{p+1}+1}\right)\right], \quad j = 0,1,\ldots,N.$$

Then $|J^j_{x,p}| = (1/2N) \sum_{k=p}^{\infty} 2N(a_k - a_{k+1}) = a_p \leq 1/(2N+1)^{j_p} = |I_{x,p}|$ and

$B(G_N;C_N) \subset \bigcup_{x\in \mathfrak{a}_p} \bigcup_{j=0}^{N} (I_{x,p} \times J^j_{x,p})$. Therefore $B(G_N;C_N)$ is contained in $(N+1)\cdot(N+1)^{jp}$ squares, each of them of dimension $1/(2N+1)^{jp}$. Hence G_N is $E(N+1)$ on C_N.

b) We show that for $a_k = 1/(2N+1)^{j_k}$ and $j_k-2 > 2(j_{k-1}+1)$, G_N is $E(N)$ on no portion of C_N. Let K be a portion of C_N and let $n > 2$ be a positive integer such that, if

$$I' = \left[\sum_{i=1}^{j_n} \frac{c_i}{(2N+1)^i} , \sum_{i=1}^{j_n} \frac{c_i}{(2N+1)^i} + R_{j_n+1} \right], \text{ then } K \supset K' = I' \cap C_N.$$

We have $|\varphi_N(K')| = 1/(2N+1)^{j_n}$. We show that G_N is not $E(N)$ on K'. Let $I = [a,b]$ be a closed interval, $a,b \in K'$. Then $I \cap C_N = I \cap K'$. We claim that if $G_N(I \cap K') \subset \bigcup_{i=1}^{N} J_i$, then

$$(1) \qquad |\varphi_N(I)| < \sum_{i=1}^{N} |J_i|.$$

Let $\{I_k\}$ be a sequence of nonoverlapping closed intervals such that $K' \subset \bigcup I_k$. Then for $D_{ki} = I_k \times J_{ki}$, with $B(G_N;K') \subset \bigcup_k \bigcup_{i=1}^{N} D_{ki}$ we have by (1) that $\sum_k \sum_{i=1}^{N} (\text{diam } D_{ki}) > \sum_k \sum_{i=1}^{N} |J_{ki}| > \sum_k |\varphi_N(I_k)| > |\varphi_N(K')| = 1/(2N+1)^{j_n}$. Hence G_N is not $E(N)$ on K'. It remains to show (1).

Let $I = [a,b]$, $a,b \in K'$. Then there exists a positive integer m such that $1/(2N+1)^{m+1} < |I| < 1/2N+1)^m$. Since $|I| < 1/(2N+1)^m$, there exist $c_1,c_2,\dots,c_m \in \{0,2,\dots,2N\}$ such that for each $x \in I \cap K'$, $c_i(x) = c_i$, $i = 1,2,\dots,m$. Since $|I| > 1/(2N+1)^{m+1}$ we have four possible cases:

1) $c_{m+1}(a) = c_{m+1}(b) = c_{m+1}$;

2) $c_{m+1}(b) - c_{m+1}(a) > 4$;

3) $c_{m+1}(a) + 2 = c_{m+1}(b)$ and $b-a = 1/(2N+1)^{m+1}$;

4) $c_{m+1}(a) + 2 = c_{m+1}(b)$ and $b-a > 1/(2N+1)^{m+1}$.

1) We have $a = \sum_{i=1}^{m+1} \frac{c_i}{(2N+1)^i}$ and $b = a + R_{m+2}$. Now the proof is similar to 2).

2) Let $c_{m+1} = c_{m+1}(a) + 2$, $A = \sum_{i=1}^{m+1} \frac{c_i}{(2N+1)^i}$ and $B = A + R_{m+2}$.

Then $[a,b] \supset [A,B]$ and $B(G_N; I \cap K') \supset B(G_N;[A,B] \cap K')$. Let k be a positive integer such that $j_k < m+2 \leq j_{k+1}$ and let

$$D_j = G_N(A + \frac{2j}{(2N+1)^{j_{k+1}}}) \quad \text{and} \quad E_j = G_N(A + \frac{2j}{(2N+1)^{j_{k+1}}} + R_{j_{k+1}+1}).$$

Then $G_N([A,B] \cap K') \subset \bigcup_{j=0}^{N} [D_j,E_j]$ and $D_{j+1} > E_j$, $j = 0,1,\dots,N-1$

(This fact will be shown below.) If we cover the set $G_N([A,B] \cap K')$ with N intervals $J_1,J_2,\dots,J_N$ at least one of these intervals contains an interval $[E_j,D_{j+1}]$ for some $j \in \{0,1,\dots,N-1\}$. Hence

$$\sum_{i=1}^{N} |J_i| \geq D_{j+1} - E_j = \frac{a_{k-1}-a_k}{N} - a_k =$$

$$= \frac{a_{k-1}}{N} (1 - \frac{N+1}{(2N+1)^{j_k-j_{k-1}}}) > \frac{a_{k-1}}{N} (1 - \frac{N+1}{2N+1}) = \frac{a_{k-1}}{2N+1} .$$

Therefore we have

$$\sum_{i=1}^{N} |J_i| > \frac{a_{k-1}}{2N+1} . \tag{2}$$

Let $I_1 = [\sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} , \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} + R_{m+1}]$. Then $I \subset I_1$ and

$|\varphi_N(I)| \leq |\varphi_N(I_1)| = 1/(N+1)^m \leq 1/(N+1)^{j_k-2} \leq 1/(N^2+2N+1)^{j_{k-1}+1} < a_{k-1}/(2N+1)$.

By (2) we easily have (1).

3) Let $c_{m+1} = c_{m+1}(a)$. Then $a = \sum_{i=1}^{m+1} \frac{c_i}{(2N+1)^i} + R_{m+2}$ and

$b = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} + \frac{c_{m+1}+2}{(2N+1)^{m+1}}$. Hence $\varphi_N(a) = \varphi_N(b)$. Now (1) follows easily.

4) Let $c_{m+1} = c_{m+1}(a)$, $A = \sum_{i=1}^{m+1} \frac{c_i}{(2N+1)^i} + R_{m+2}$ and $B = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} + \frac{c_{m+1}+2}{(2N+1)^{m+1}}$. Then $a = \sum_{i=1}^{m+1} \frac{c_i}{(2N+1)^i} + \sum_{i=m+2}^{\infty} \frac{c_i(a)}{(2N+1)^i}$ and $b = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} + \frac{c_{m+1}+2}{(2N+1)^{m+1}} + \sum_{i=m+2}^{\infty} \frac{c_i(b)}{(2N+1)^i}$. Now we have two possible situations:

(i) $A-a < b-B$ and $b \neq B$;

(ii) $A-a > b-B$ and $a \neq A$;

(i) Since $b \neq B$, there exists a positive integer p such that $p = \inf\{i \in N : i > m+2,\ c_i(b) > 2\}$. Then $p > m+2$ and $B(G_N; I \cap K') \supset B(G_N; [B, B + R_{p+1}] \cap K')$. Let k be a positive integer such that $j_k < p+1 < j_{k+1}$. By analogy with case 2), if we cover the set $G_N([B, B + R_{p+1}) \cap K')$ with N intervals $J_1, \ldots, J_N$, then $\sum_{i=1}^{N} |J_i| > \frac{a_{k-1}}{2N+1}$. We have $b-B < R_p = 1/(2N+1)^{p-1}$. Let $A_2 =$ $[A - R_p,\ B_2 = B + R_p]$ and $I_2 = [A_2, B_2]$. Then $I \subset I_2$ and $|\varphi_N(I_2)| = 2/(N+1)^{p-1} < 1/(N+1)^{j_k-2} < a_{k-1}/(2N+1)$. Hence we obtain (1).

(ii) Since $a \neq A$, there exists a positive integer p such that $p = \inf\{i \in N : i > m+2,\ c_i(a) < 2N-2\}$. Clearly $p > m+2$. Let $A_3 = A - R_{p+1} = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i} + \sum_{i=m+2}^{p} \frac{2N}{(2N+1)^i}$. Then $[a, A] \supset [A_3, A]$ and $A = A_3 + R_{p+1}$. Therefore $B(G_n; I \cap K') \supset B(G_n; [A_3, A_3 + R_{p+1}] \cap K')$. Now by analogy with (i) we obtain (1).

(c) We show that G_N is $A(N^2+2N+1)$ on C_N for $a_k < 1/(2N+1)^{j_k}$. Let $I = [a,b]$ be an interval, $a,b \in C_N$, such that $I \cap C_N \neq \emptyset$ and $1/(2N+1)^{m+1} < |I| < 1/(2N+1)^m$, for some positive integer m. Since $|I| < 1/(2N+1)^m$, there exist $c_i \in \{0,2,\ldots,2N\}$, $i = 1,\ldots,m$ such that, if $x \in I \cap C_N$, $c_i(x) = c_i$, $i = 1,2,\ldots,m$. We may suppose without loss of generality that $m > j_2$. Let $A_1 = \sum_{i=1}^{m} \frac{c_i}{(2N+1)^i}$ and $B_1 = A_1 + R_{m+1}$. Let k be the first positive integer such that $j_k > m+1$, and let $\mathfrak{a} = \{x \in C_N : x = \frac{c_{j_k}(x)}{(2N+1)^{j_k}} + \frac{c_{j_{k+1}}(x)}{(2N+1)^{j_{k+1}}}\}$. Then $\mathfrak{a}$ has (N^2+2N+1) elements. For each $x \in \mathfrak{a}$, let $J_x = [G_N(A_1+x), G_N(A_1+x+R_{j_{k+1}+1})]$. Then $G_N(I \cap C_N) \subset G_N([A_1,B_1] \cap C_N) \subset \bigcup_{x\in\mathfrak{a}} J_x$ and $|J_x| = (1/2N) \sum_{i=k}^{\infty} 2N(a_i-a_{i+1}) = a_k < 1/(2N+1)^{j_k} < 1/(2N+1)^{m+1} < |I|$. Hence $\sum_{x\in\mathfrak{a}} |J_x| < (N^2+2N+1)\cdot|I|$ and G_N is $A(N^2+2N+1)$ on C_N.

d) We show that G_N is $B(N^2+2N)$ on no portion of C_N, for $a_k = 1/(2N+1)^{j_k}$ and $j_{k+2} - j_k > 2j_{k+1} + 2$, $k = 0,1,2,\ldots$. Let K be a portion of C_N. Then there exist $c_i \in \{0,2,\ldots,2N\}$, $i = 1,2,\ldots,j_p-1$, such that K contains the set $K_1 = C_N \cap [S_p, S_p+R_{j_p}]$, where $S_p = \sum_{i=1}^{j_p-1} \frac{c_i}{(2N+1)^i}$. We show that G_N does not satisfy $B(N^2+2N)$ on K_1. Let $p \in N$, $p > 2$ and

$$\mathfrak{a}_p = \left\{x \in C_N : x = \sum_{i=p}^{j_{p+2}-1} \frac{c_i(x)}{(2N+1)^i}\right\}.$$

For each $x \in \mathfrak{a}_p$ let $I_{p,x} = [S_p+x, S_p+x+R_{j_{p+2}}]$. Clearly $I_{p,x} \cap K_1 \neq \emptyset$ and $\mathfrak{a}_p$ had $(N+1)^{j_{p+2}-j_p}$ elements. Let

$$\mathfrak{B}_p = \left\{y \in C_N : y = \frac{c_{j_{p+2}}(y)}{(2N+1)^{j_{p+2}}} + \frac{c_{j_{p+3}}(x)}{(2N+1)^{j_{p+3}}}\right\}.$$

Clearly $\mathfrak{B}_p$ has (N^2+2N+1) elements, namely $y_1 < y_2 < \cdots < y_{N^2+2N+1}$. For each $x \in \mathfrak{a}_p$

and $y \in \mathcal{B}_p$ let $A_{x,y} = G_N(S_p+x+y)$ and $B_{x,y} = G_N(S_p+x+y+R_{j_{p+3}+1})$. We have

(3) $$G_N(I_{p,x} \cap C_N) \subset \bigcup_{y\in\mathcal{B}_p} [A_{x,y}, B_{x,y}].$$

Let $y,z \in \mathcal{B}_p$, $y < z$. Then we have two possible situations:

1. $c_{j_{p+2}}(y) < c_{j_{p+2}}(z)$;

2. $c_{j_{p+2}}(y) = c_{j_{p+2}}(z)$ and $c_{j_{p+3}}(y) < c_{j_{p+3}}(z)$.

1. We have $A_{x,z} - B_{x,y} > (2/2N)\cdot(a_p - a_{p+1}) - (1/2N)\cdot \sum_{i=p+1}^{\infty} 2N(a_i - a_{i+1}) > (1/N)(a_p - a_{p+1}) - a_{p+1} = T_p$, where

$$T_p = \frac{a_p-(N+1)a_{p+1}}{N}.$$

2. Analogously to 1., we obtain $A_{x,z} - B_{x,y} > T_{p+1}$.

Since $T_p > T_{p+1}$, we have in both cases

(4) $A_{x,z} - B_{x,y} > T_{p+1}$.

By (3) and (4), if we cover $G_N(I_{p,x} \cap C_N)$ wtih (N^2+2N) intervals $J_{x,i}$, $i = 1,2,\ldots,N^2+2N$, then there exists at least one $y_i \in \mathcal{B}_p$ such that at least one of the intervals $J_{x,i}$ contains the interval $[B_{x,y_i}, A_{x,y_{i+1}}]$. Hence

$$\sum_{x\in\mathcal{a}_p} \sum_{i=1}^{N^2+2N} |J_{x,i}| > (N+1)^{j_{p+2}-j_p}\, T_{p+1} > (N^2+2N+1)^{j_{p+1}+1}\cdot(1/N)\cdot \left(\frac{1}{(2N+1)^{j_{p+1}}} - \frac{N+1}{(2N+1)^{j_{p+1}+1}}\right) = \left[\frac{N^2+2N+1}{2N+1}\right]^{j_{p+1}+1} \underset{p \longrightarrow \infty}{\longrightarrow} \infty.$$

<u>Theorem</u> 2. <u>There exists a continuous function</u> F <u>on</u> $[0,1]$ <u>such that</u>: a) F <u>is</u> $\overline{N}$ <u>on</u> C; b) <u>for each positive integer</u> N, F <u>is</u> $E(N)$ <u>on no portion of</u> C.

Proof. For each $x \in C$, let $F(x) = \sum_{i=1}^{\infty} \frac{c_{2i}(x)}{3^i}$. Then F is continuous on C. Extending F linearly on each interval contiguous to C we have F defined and continuous on $[0,1]$.

a) Let $p \in N$ and $\mathcal{A}_p = \left\{x \in C : x = \sum_{i=1}^{2p} \frac{c_i(x)}{3^i}\right\}$. Then $\mathcal{A}_p$ has 2^{2p} elements. For each $k = 1,2,3,\dots$ let $R'_k = \sum_{i=k}^{\infty} 2/3^i$ and for $x \in \mathcal{A}_p$ let $I_{p,x} = [x, x+R'_{2p+1}]$. Let $\mathcal{B}_p = \left\{y \in C : y = \sum_{i=p+1}^{2p} \frac{c_{2i}(y)}{3^{2i}}\right\}$.

Then $\mathcal{B}_p$ has 2^p elements. For each $x \in \mathcal{A}_p$ and $y \in \mathcal{B}_p$ let $J_{p,x,y} = [F(x+y), F(x+y+R'_{4p+1})]$. Then $|I_{p,x}| = |J_{p,x,y}| = 1/3^{2p} = 1/9^p$. Hence $B(F;C) \subset \bigcup_{x\in\mathcal{A}_p} \bigcup_{y\in\mathcal{B}_p} (I_{p,x} \times J_{p,x,y})$, $p \in N$. Therefore $B(F;C)$ is contained in $2^{2p}\cdot 2^p = 8^p$ squares, each of them of dimension $1/9^p$. Now it follows easily that F is $\overline{N}$ on C.

b) Let K be a portion of C and

$I' = [\sum_{i=1}^{p} c_i/3^i, \sum_{i=1}^{p} c_i/3^i + R'_{p+1}]$, $p \in N$. Then $K \supset K' = I' \cap C$, for some $p \in N$. We show that F is not $E(2^q-1)$ on K', $q \in N$. We may suppose without loss of generality that $p > 8q+13$. Let $I = [a,b]$, $a,b \in K'$. Then $I \cap C = I \cap K'$. We claim that if $F(I \cap K') \subset \bigcup_{i=1}^{2^q-1} J_i$, then

$$(5) \qquad |\varphi(I)| < \sum_{i=1}^{2^q-1} |J_i|.$$

Let $\{I_k\}$ be a sequence of nonoverlapping closed intervals such $K' \subset \bigcup I_k$. Then for $D_{ki} = I_k \times J_{ki}$ with $B(F;K') \subset \bigcup_k \bigcup_{i=1}^{2^q-1} D_{ki}$, we have by (5) that

$\sum_k \sum_{i=1}^{2^q-1} (\operatorname{diam} D_{ki}) \geq \sum_k \sum_{i=1}^{2^q-1} |J_{ki}| \geq \sum_k |\varphi(I_k)| \geq |\varphi(K')| = 1/2^p$. Hence F is not $E(2^q-1)$ on K'. It remains to prove (5).

Let $I = [a,b]$, $a,b \in K'$. Then there exists a positive integer m such that $1/3^{m+1} \leq |I| < 1/3^m$. Since $|I| < 1/3^m$, it follows that there exist $c_1, c_2, \ldots, c_m \in \{0,2\}$ such that for each $x \in I \cap C$, $c_i(x) = c_i$, $i = 1,2,\ldots,m$. Since $|I| \geq 1/3^{m+1}$ we have three possible situations:

1. $c_{m+1}(a) = c_{m+1}(b) = c_{m+1}$;

2. $c_{m+1}(a) = 0$, $c_{m+1}(b) = 2$ and $b-a = 1/3^{m+1}$;

3. $c_{m+1}(a) = 0$, $c_{m+1}(b) = 2$ and $b-a > 1/3^{m+1}$.

1. We have $a \sum_{i=1}^{m+1} c_i/3^i$, $b = a + R'_{m+2}$ and

(6) $\varphi(b) - \varphi(a) = 1/2^{m+1}$.

Let n be the first positive integer such that $m+2 \leq 2n$, and let $\mathcal{a}_{nq} = \left\{x \in C : x = \sum_{i=1}^{q} \frac{c_{2n+2i}}{3^{2n+2i}}\right\}$. Then $\mathcal{a}_{nq}$ has 2^q elements; namely $x_1 < x_2 < \cdots < x_{2^q}$. Let $A_x = F(a+x)$ and $B_x = F(a+x+R'_{2n+2q+1})$, $x \in \mathcal{a}_{nq}$. We have

(7) $F([a,b] \cap C) \subset \bigcup_{x \in \mathcal{a}_{nq}} [A_x, B_x]$ and

(8) $A_y - B_x \geq 1/3^{n+q}$, $x,y \in \mathcal{a}_{nq}$, $x < y$.

Indeed, let $k \in \{1,2,\ldots,q\}$ such that $c_{2n+2j}(x) = c_{2n+2j}(y)$, $j = 1,2,\ldots,k-1$, $c_{2n+2k}(x) = 0$ and $c_{2n+2k}(y) = 2$. Then $A_y - B_x \geq 2/3^{n+k} - R'_{k+1} = 1/3^{n+k} \geq 1/3^{n+q}$ and we have (8).
By (7) and (8), if we cover $F([a,b] \cap C)$ with 2^q-1 intervals J_i, $i = 1,2,\ldots,2^q-1$, then at least one of them contains an interval $[B_{x_i}, A_{x_{i+1}}]$ for some $i \in \{1,2,\ldots,2^q-1\}$. Hence

$$(9)\qquad \sum_{i=1}^{2^q-1} |J_i| \geq 1/3^{n+q}\ .$$

Clearly $2n-2 \leq m+1 \leq 2n-1$. Since $m+1 \geq p > 8q+13$, it follows that $2n \geq m+2 \geq p+1 \geq 8q+14$. Hence $n \geq 4q+7$. We have

$$(10)\qquad \frac{2^{m+1}}{3^{n+q}} \geq \frac{2^{2n-2}}{3^{n+q}} = (1/4)\cdot(4/3)^n\cdot(1/3)^q \geq (1/4)\cdot((4/3)^n\cdot(1/3))^q\cdot(4/3)^7 > 1.$$

Then (5) follows by (6), (9) and (10).

2. We have $a = \sum_{i=1}^{m} c_i/3^i + R'_{m+2}$, $b = \sum_{i=1}^{m} c_i/3^i + 2/3^{m+1}$ and $\varphi(a) = \varphi(b)$. Hence (5) follows.

3. Let $A = \sum_{i=1}^{m} c_i/3^i + R'_{m+2}$ and $B = \sum_{i=1}^{m} c_i/3^i + 2/3^{m+1}$. Then

$$a = \sum_{i=1}^{m} c_i/3^i + \sum_{i=m+2}^{\infty} c_i(a)/3^i \quad\text{and}\quad b = \sum_{i=1}^{m} c_i/3^i + 2/3^{m+1} + \sum_{i=m+2}^{\infty} c_i(b)/3^i.$$

Now we have two possibilities:

(i) $A-a \leq b-B$ and $b \neq B$;

(ii) $A-a \geq b-B$ and $a \neq A$.

(i) Since $b \neq B$, there exists a positive integer s such that $s = \inf\{i \in N: i \geq m+2,\ c_i(b) = 2\}$. Then $s \geq m+2$ and $B(F;I \cap K') \supset B(F;[B,\ B + R'_{s+1}] \cap K')$. Let n be a positive integer such that $2n-1 \leq s+1 \leq 2n$. Then $\varphi(I) \subset \varphi([A-R'_s,\ B+R'_s])$ and

$$(11)\qquad |\varphi(I)| \leq 2/2^{s-1}.$$

If we cover the set $F([B,\ B+R'_{s+1}] \cap K')$ with w^q-1 intervals J_i, $i = 1,2,\ldots,2^q-1$, then

$$(12)\qquad \sum_{i=1}^{2^q-1} |J_i| \geq 1/3^{n+q}.$$

Clearly $2n \geq s+1 \geq m+3 \geq p+2 \geq 8q+15$. Hence $n > 4q+8$ and

$$(13)\quad \frac{2^{s-2}}{3^{n+q}} \geq \frac{2^{2n-4}}{3^{n+q}} = (1/16)\cdot(4/3)^{n}\cdot(1/3)^{q} \geq (1/16)\cdot((4/3)^{4}\cdot(1/3))^{q}\cdot(4/3)^{8} > 1.$$

By (11), (12) and (13) we have (5).

(ii) Since $a \neq A$, there exists a positive integer s such that $s = \inf\{i \in N : i \geq m+2,\ c_i(a) = 0\}$. Clearly $s \geq m+2$. Let $A_1 = A - R'_{s+1} = \sum_{i=1}^{m} c_i:3^i + \sum_{i=m+2}^{s} 2/3^i$. Then $[a,A] \supset [A_1,A]$ and $A = A_1 + R'_{s+1}$. Therefore $B(F;I \cap K') \supset B(F;[A_1,\ A_1+R'_{s+1}] \cap K')$. By analogy with (i), (5) follows.

We are indebted to Professor Solomon Marcus for his help in preparing this article.

<u>References</u>

1. Ene, V.: A Study of Foran's Conditions A(N), B(N) and his Class $\mathcal{F}$. Real Analysis Exchange, 10 (1985), 194-211.

2. Ene, G. and Ene, V.: Nonabsolutely Convergent Integrals. Real Analysis Exchange, 11 (1985-86), 121-133.

3. Foran, J.: An Extension of the Denjoy Integral. Proc. Amer. Math. Soc., 49 (1975), 359-365.

4. Foran, J.: On functions whose graph is of linear measure 0 on sets of measure 0. Fund. Math. XCVI (1977), 31-36.

5. Saks, S.: Theory of the Integral. 2nd, rev. ed. Mongrafie Matematyczne, vol. VII, PWN, Warsaw (1937).

Received December 30, 1985.