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Continuous Linear Transformations of n-Convex Functions

1. Introduction. In a recent paper [1] Bojanic and Roulier obtained a result on the uniform approximation of n-convex functions by n-convex splines and, as an application, proved necessary and sufficient conditions that a continuous linear operator transforms every continuous, n-convex function into an r-convex function.

Their proof is based on the fact that the Bernstein polynomials of a continuous, n-convex function F (which converges uniformly to F) are themselves n-convex.

In this paper we express the result of Bojanic and Roulier in a slightly different notation and terminology and then we take a similar approach, after obtaining various decompositions of an n-convex function, to obtain new sufficient conditions that a continuous linear operator transform every continuous n-convex function into an r-convex function.

2. n-convex functions and n-convex splines. Let $F(x)$ be defined on $[a,b]$. The n^{th} divided difference of F is defined by

$$V_n(F;x_k) \equiv \sum_{k=0}^{n} F(x_k)/w_n'(x_k),$$

where

$$a \le x_0 < x_1 < \dots < x_{n-1} < x_n \le b$$

and

$$w_n(x) = (x-x_0)(x-x_1)\dots(x-x_n).$$

Definition 2.1. If $V_n(F;x_k) \ge 0$ for all choices of $(n+1)$ distinct points $x_0, x_1, \dots, x_n$ in $[a,b]$, then F is said to be n-convex on $[a,b]$.

The set of all continuous, n-convex functions on $[a,b]$ will be denoted by $K_n[a,b]$, and the set of all functions on $[a,b]$, by $L[a,b]$.

It is easy to see that all polynomials of degree $(n-1)$ and those polynomials of degree n with positive leading coefficient are n-convex.

Some important properties of n-convex functions are contained in the following theorem.

Theorem 2.1. If F is n-convex on $[a,b]$, then

(a) $F^r(x)$ exists and is continuous, $1 \le r \le n-2$, $x \in (a,b)$;

(b) $F^{(n-1)}(x)$ exists except possibly on a countable set in (a,b) and is increasing;

(c) If $x \in [\alpha,\beta] \subset (a,b)$, $F(x)$ is the $(n-1)^{th}$ integral of $F^{(n-1)}(x)$, i.e.

$$F(x) = \frac{1}{(n-2)!} \int_\alpha^x (x-t)^{n-2} F^{(n-1)}(t)\,dt + \sum_{k=0}^{n-2} F^{(k)}(\alpha) \frac{(x-\alpha)^k}{k!} .$$

Proof: This follows from Theorem 7 and Corollary 8 of [2] (where the results are erroneously stated to hold on the closed interval $[a,b]$.)

However, the Bernstein polynomials provide n-convex *uniform approximants* to a *continuous* n-convex function which satisfy (a), (b), and (c) on the *closed* interval $[a,b]$. As we shall see, even more is true.

Definition 2.2. *If* $a < c_1 < c_2 < \dots < c_m < b$, *and if* x_+^n denotes the function

$$x_+^n = \begin{cases} x^n & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

then a spline function of degree n, $n \geq 1$, *with knots* $c_1, c_2, \dots, c_m$ *is defined to be any function of the form*

$$g(x) = \sum_{r=0}^{n} \alpha_r x^r + \sum_{j=1}^{m} \beta_j (x-c_j)_+^n .$$

Theorem 2.2. [3] *A spline function of degree* n *is any* n^{th} *order indefinite integral of a step function*.

It is easy to show that if the step function of Theorem 2.2 is increasing, then the spline function of degree n is $(n+1)$ convex.

We now state the two theorems of Bojanic and Roulier mentioned earlier and prove a third in the spirit of the first.

Theorem 2.3. (Theorem 1,[1]). *Every* $F \in K_n[a,b]$, $n \geq 2$ *can be approximated uniformly on* $[a,b]$ *by spline functions of the form*

$$\psi_{m,N}(F,x) = \sum_{k=0}^{n-1} \binom{n}{k} \left(\Delta^k_{\frac{b-a}{m}} F(a) \right) (x-a)^k/(b-a)^k$$

$$+ \frac{C(F,m)}{N} \sum_{k=1}^{N-1} (x-c_k)_+^{n-1}/(b-a)^{n-1}$$

where $C(F,m) = \binom{m}{n-1} \sum_{k=0}^{m-n} \Delta^n_{\frac{b-a}{m}} F\left(a+k\left(\frac{b-a}{m}\right)\right) \geq 0$, $m \geq n$, $N \geq 2$, $c_k \in (a,b)$, $k = 1,2,\ldots,N-1$, *and*

$$\Delta_h^k F(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} F(x+rh).$$

Theorem 2.4. (Theorem 2.[1]). *Let* $T: C[a,b] \to L[a,b]$ *be a continuous linear operator. In order that for every* $F \in K_n[a,b]$, $n \geq 2$, *we have* $T(F) \in K_r[a,b]$, $r \geq 0$, *it is necessary and sufficient that*

(i) $V_r(T(P);x_k) = 0$ *for every polynomial* P *of degree* $\leq n-1$ *and every set of* $r+1$ *points* $x_0 < x_1 < \ldots < x_r$ *in* $[a,b]$;

(ii) $T\left[(t-c)_+^{n-1}\right] \in K_r[a,b]$ _for every_ $c \in (a,b)$.

Theorem 2.5. If $F \in K_n[a,b]$, _then given_ $\epsilon > 0$ _there exists_ $G \in K_n[a,b]$ _such that_

(a) $G^{(n)}(x)$ _exists in_ $[a,b]$;

(b) $$G(x) = \sum_{k=0}^{n-2} G^{(k)}(a) \frac{(x-a)^k}{k!} + \int_a^x \cdots \int_a^{x_1} G^{(n-1)}(t)\,dt\,dx_1 \ldots dx_{n-2}$$

where $G^{(n-1)}$ _is increasing on_ $[a,b]$;

(c) $|G(x)-F(x)| < \epsilon$, _for_ $x \in [a,b]$.

Proof: There is no loss in generality in assuming $[a,b] = [0,1]$. Let $G(x)$ be the Bernstein polynomial of F of degree $m \geq n$:

$$G(x) = \sum_{k=0}^{m} \binom{m}{k} F\left(\frac{k}{m}\right) x^k (1-x)^{m-k} ,$$

where m is so large that $|G(x)-F(x)| < \epsilon$ for $x \in [0,1]$. It is shown in [4] that since $F \in K_n[0,1]$, $G^{(n)}(x) \geq 0$, for $x \in [0,1]$ and (b) follows.

The class of functions $G \in K_n[a,b]$ which satisfy (a) and (b) of Theorem 2.5 will be denoted by $\bar{K}_n[a,b]$.

3. The uniform approximation of continuous n-convex functions by n-convex splines. It is easily shown that if a function $f(x)$ is continuous and monotonic on $[0,1]$, then it can be uniformly approximated on $[0,1]$ by step functions of the form

$$f_N(x) = f(0) + \left(\frac{f(1)-f(0)}{N}\right) \sum_{k=1}^{N-1} (x-c_k)_+^0 ,$$

where $N \geq 2$, $0 < c_1 < c_2 < \ldots < c_{N-1} < b$, and

$$(x-c)_+^0 = \begin{cases} 0 & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c . \end{cases}$$

We now state and prove the theorem on the approximation of n-convex functions by n-convex splines.

Theorem 3.1. If $F \in K_n[a,b]$, then F can be uniformly approximated by n-convex spline functions of the form

$$G_N(x) = \sum_{k=0}^{n-1} \left(\frac{\alpha_k (x-a)^k}{k!}\right) + \left(\frac{C}{N}\right) \sum_{j=1}^{N-1} (x-c_j)_+^{n-1} , \qquad (3.1)$$

where $C > 0$, $N \geq 2$, and $a < c_1 < c_2 < \ldots < c_{N-1} < b$.

(Note: This has the same form as the corresponding decomposition in [1] without the constants precisely determined.)

Proof: (For [a,b] = [0,1]). The first step will be to approximate F by $G \in \bar{K}_n[0,1]$. Thus, writing $g(t) = G^{(n-1)}(t)$ we have by Theorem 2.5

$$G(x) = \sum_{k=0}^{n-2} G^{(k)}(0)\, \frac{x^k}{k!} + \int_0^x \cdots \int_0^{x_1} g(t)\,dt\,dx_1 \ldots dx_{n-2},$$

where $g(t)$ is continuous and increasing. Now let

$$g_N(x) = g(0) + \left(\frac{g(1)-g(0)}{N}\right) \sum_{k=1}^{N-1} (x-c_k)_+^0$$

be an approximating spline to the continuous, increasing function $g(t)$. Then define G_N by

$$G_N(x) = \sum_{k=0}^{n-2} \frac{G^{(k)}(0)x^k}{k!} + \int_0^x \cdots \int_0^{x_1} g_N(t)\,dt\,dx_1 \ldots dx_{n-2} .$$

Then $G_N(x)$ is a convex spline which after integration becomes

$$G_N(x) = \sum_{k=0}^{n-1} \frac{G^{(k)}(0)x^k}{k!} + \left(\frac{g(1)-g(0)}{N}\right) \sum_{j=1}^{N-1} (x-c_j)_+^{n-1} ,$$

which is the decomposition required in (3.1).

Furthermore, it is clear that given $F \in K_n[a,b]$ and $\epsilon > 0$ we may choose $G_N(x)$ so that

$$|F(x)-G_N(x)| \le |F(x)-G(x)| + |G(x)-G_N(x)|$$
$$< \epsilon + \frac{\epsilon\, x^{n-1}}{(n-1)!} < 2\epsilon .$$

The proof of Theorem 2.4 follows as in [1].

4. Sufficient Conditions for Convexity Preserving Linear Operators.

We first obtain a decomposition of $G \in \bar{K}_n[a,b]$ which is interesting by itself.

Theorem 4.1. *If* $G \in \bar{K}_n[a,b]$, *then for* $x \in [a,b]$

$$G(x) = H(x) + \sum_{k=0}^{n-1} G^{(k)}(a) \frac{(x-a)^k}{k!}, \qquad (4.1)$$

where $H(x)$ *is increasing and* $H(x) \in \bar{K}_r[a,b]$ *for each fixed* r, $2 \le r \le n$.

Proof: Let H be defined by

$$H(x) \equiv G(x) - \sum_{k=0}^{n-1} G^{(k)}(a) \frac{(x-a)^k}{k!}.$$

Then $H(x)$ is n-convex and $H^{(k)}(a) = 0$, $k = 0,1,2,\dots,n-1$, and $H^{(n-1)}(x)$ is increasing. We have then for $x \in [a,b]$, $(H^{(n-2)}(x))' = H^{(n-1)}(x) \ge H^{(n-1)}(a) = 0$, and it follows that $H^{(n-2)}(x)$ is increasing on $[a,b]$ and $H(x) \in \bar{K}_{n-1}[a,b]$.

Repeating the same process on $H^{(n-2)}(x)$ we obtain $(H^{(n-3)}(x))' = H^{(n-2)}(x) \ge H^{(n-2)}(a) = 0$ and so $H^{(n-3)}(x)$ is increasing and $H(x) \in K_{n-2}[a,b]$. Continuing in this way we obtain after $(n-1)$ steps that $H'(x) \geq 0$ and $H(x)$ is increasing on $[a,b]$.

Theorem 4.2. *If* $F \in K_n[a,b]$, *then* F *can be uniformly approximated by splines of the form*

$$G_{N_s}(x) = \sum_{k=0}^{n-1} \frac{\alpha_k (x-a)^k}{k!} + \left(\frac{C_s}{N_s}\right) \sum_{j=1}^{N_s - 1} (x-c_j)_+^{s-1}, \qquad (4.2)$$

for each $1 \leq s \leq n-1$, *where* $C_s > 0$ *and* $N_s > 0$ *are constants (depending on* s), *and*

$$a < c_1 < c_2 < \ldots < c_{N_s - 1} < b.$$

Proof: (For fixed s and $[a,b] = [0,1]$). As before we first approximate F by $G \in \bar{K}_n[0,1]$. Then, utilizing the decomposition in (4.1) and writing $g(t) = H^{(s-1)}(t)$ (noting that $H^{(k)}(0) = 0$, $1 \leq k \leq n-1$) we next obtain

$$\begin{aligned} H(x) &= \int_0^x \cdots \int_0^{x_1} g(t)\,dt\,dx_1 \ldots dx_{s-2} \\ &= \left(\frac{C_s}{N_s}\right) \sum_{j=1}^{N_s - 1} (x-c_j)_+^{s-1}, \end{aligned}$$

as in the proof of Theorem 3.1, and (4.2) follows.

We now state and prove our result on continuous linear transformations of n-convex functions.

Theorem 4.3. Let $T: C[a,b] \to L[a,b]$ be a continuous linear operator. In order that for every $F \in K_n[a,b]$, $n \geq 2$, we have $T(F) \in K_r[a,b]$, $r \geq 0$, it is sufficient that

(i) $V_r(T(P);x_k) = 0$ for every polynomial P of degree $\leq n-1$ and every set of $r+1$ points $x_0 < x_1 < \ldots < x_r$ in $[a,b]$;

(ii) $T((t-c)_+^{s-1}) \in K_r[a,b]$ for some fixed s, $2 \leq s \leq n$, and every $c \in (a,b)$.

Proof: (For fixed s). Let $f \in K_n[a,b]$ and let $G_{N_s}(x)$ be a spline function of the form (4.2). Since $C_s > 0$ and $N_s > 0$, and T is linear, conditions (i) and (ii) imply that $V_r(T(G_{N_s});x_k) \geq 0$ for every set of $r+1$ points $x_0 < x_1 < \ldots < x_r$ in $[a,b]$. Since T is continuous and G_{N_s} approximates F uniformly it follows that

$$V_r(T(F);x_k) \geq 0$$

and so $T(F) \in K_r[a,b]$.

Notes: (a) The theorem as stated does not hold for $s = 1$ since $(t-c)_+^0$ is not continuous.

(b) Condition (ii), for $s < n$, is not a necessary condition since $(t-c)_+^{s-1}$ need not be n-convex.

References

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