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Michael J. Evans, Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695-8205
Paul D. Humke, Department of Mathematics, St. Olaf College, Northfield, Minnesota 55057

Approximate Continuity Points and L-points of Integrable Functions

All functions considered in this paper will be Lebesgue measurable, real valued functions defined on the real line $\mathbb{R}$. For such a function f, we let $\mathcal{A}(f) = \mathcal{A}$ denote the set of points at which f is approximately continuous, and we let $\mathcal{L}(f) = \mathcal{L}$ denote the set of L-points of f, where x is called an L-point provided

$$\lim_{h\to 0} \frac{1}{h}\int_0^h f(x+t)dt = f(x).$$

The classical works of Denjoy [2] and Lebesgue [7] show that almost every point of $\mathbb{R}$ is in $\mathcal{A}$ and that if, further, f is locally integrable, then almost every point of $\mathbb{R}$ is also in $\mathcal{L}$. Denjoy [2] further showed that if f is bounded, then $\mathcal{A} \subseteq \mathcal{L}$ and that this containment is not true for all locally integrable functions. The purpose of the present paper is to further examine the relationship between the sets $\mathcal{A}$ and $\mathcal{L}$ for various classes of functions.

Throughout the paper we shall use $m(E)$ to denote the Lebesgue measure of a measurable set E, $A \setminus B$ to denote the intersection of the set A with the complement of the set B, and $A \Delta B$ to denote $(A \setminus B) \cup (B \setminus A)$.

We begin by taking a look at the category relationship between $\mathcal{A}$ and $\mathcal{L}$. While both $\mathbb{R} \setminus \mathcal{A}$ and $\mathbb{R} \setminus \mathcal{L}$ will be null sets for a locally integrable function, neither need be of the first category, even if f is bounded; indeed, a somewhat stronger claim can be made as the following easy consequence of a theorem due to Goffman [4] shows.

<u>EXAMPLE 1.</u> Let Z be any measure zero set. There is a bounded function $f: \mathbb{R} \to \mathbb{R}$ for which $Z \subseteq \mathbb{R} \setminus \mathcal{L}$; in fact, at no point of Z does

$$\lim_{h \to 0} \frac{1}{h}\int_0^h f(x + t)dt$$ exist.

PROOF. Let Z be a measure zero set. Goffman [4] has shown the existence of a measurable set S whose metric density does not exist at any point of Z. Letting f denote the characteristic function of S, we have our example.

Nonetheless, a category relationship does exist between $\mathcal{A}$ and $\mathcal{L}$ as is evidenced by the following result.

<u>THEOREM 1.</u> If $f: \mathbb{R} \to \mathbb{R}$ is a locally integrable function, then $\mathcal{A} \Delta \mathcal{L}$ is a first category set.

PROOF. We shall first show that $\mathcal{L} \setminus \mathcal{A}$ is a first category set. To this end, suppose that $\mathcal{L} \setminus \mathcal{A}$ is of the second category. For each natural number n let

$$L_n = \{x\in\mathcal{L} : \exists \text{ a set } S_x \text{ for which } \Delta^+(S_x,x) > 0 \text{ and } |f(y)-f(x)| > \frac{3}{n} \text{ for } y\in S_x\},$$

where the symbol $\Delta^+(E,t)$ denotes the upper density of the set E at the point t. Then since $\mathcal{L} \setminus \mathcal{A} = \bigcup_{n=1}^{\infty} L_n$, there is an n_0 such that L_{n_0} is a second category set.

Next, there must be an integer m for which the set $L_{n_0} \cap f^{-1}([m/n_0,(m+1)/n_0))$ is a second category set. Denote this intersection by M_{n_0}.

Finally, for each natural number k, let

$$B_k = \{x \in M_{n_0} : \left|\frac{1}{t}\int_0^t f(x + s)ds - f(x)\right| < \frac{1}{n_0} \text{ for } |t| < \frac{1}{k}\},$$

and note that since M_{n_0} is the union of the B_k's, there must exist a k_0 such that B_{k_0} is dense in some interval I, which we may assume has length less than $\frac{1}{k_0}$.

Let $x_0 \in B_{k_0}$ and without loss of generality assume that $f(x_0) = 0$. Then $S_{x_0} \cap I$ has positive measure and since almost every point of $\mathbb{R}$ is an L-point, $S_{x_0} \cap I$ must contain an L-point y of f. Since y is an L-point, there is a positive number $h < \frac{1}{k_0}$ for which

$$\left|\frac{1}{h}\int_0^h f(y + s)ds - f(y)\right| < \frac{1}{n_0},$$

and, consequently,

$$\left|\frac{1}{h}\int_0^h f(y+s)ds\right| > |f(y)| - \frac{1}{n_o} > \frac{2}{n_o},$$

where the final inequality results from the fact that $y \in S_{x_o}$.

Since B_{k_o} is dense in I, there exists a sequence $\{y_j\} \subset B_{k_o}$ such that $y_j \to y$. Now, for each j we have

$$\left|\frac{1}{h}\int_0^h f(y_j+s)ds - f(y_j)\right| < \frac{1}{n_o},$$

and hence

$$\left|\frac{1}{h}\int_0^h f(y_j+s)ds\right| \le \frac{2}{n_o}.$$

Allowing j to tend to infinity, this produces a contradiction to the last inequality of the previous paragraph and completes the proof that $\mathcal{L} \setminus \mathcal{A}$ is a first category set.

We have yet to show that $\mathcal{A} \setminus \mathcal{L}$ is a first category set. Again, suppose that it is not. For each natural number n let

$$A_n = \{x \in \mathcal{A} : \exists \text{ a sequence of numbers } h_j(x) \text{ tending to } 0 \text{ for which } \left|\frac{1}{h_j(x)}\int_0^{h_j(x)} f(x+t)dt - f(x)\right| > \frac{2}{n}\}.$$

Then, since $\mathcal{A} \setminus \mathcal{L}$ is the union of the A_n's, there must be a natural number n_o for which A_{n_o} is of second category. Furthermore, there must exist an integer m for which the set $P_{n_o} \equiv A_{n_o} \cap f^{-1}([m/n_o, (m+1)/n_o))$ is of second category.

For each natural number k let

$$C_k = \{x \in P_{n_o} : \exists \text{ a set } T_x \text{ such that for any interval } J \text{ of length less than } \frac{1}{k} \text{ containing } x, \frac{m[T_x \cap J]}{m(J)} > \frac{1}{2} \text{ and } |f(y) - f(x)| < \frac{1}{n_o} \text{ for } y \in T_x \}.$$

Since P_{n_o} is the union of the C_k's, there must be a natural number k_o for which C_{k_o} is dense in some interval I of length less than $\frac{1}{k_o}$.

Let $x_o \in C_{k_o}$ and without loss of generality assume that $f(x_o) = 0$. Since $x_o \in A_{n_o}$, there is a positive $h_o < \frac{1}{k_o}$ such that

$$\left|\frac{1}{h_o}\int_0^{h_o} f(x_o + t)dt\right| > \frac{2}{n_o}.$$

Consequently, the set $E = \{y: |f(y)| > \frac{2}{n_o}\}$ has positive measure in I. Let y_o be a point of density of $E \cap I$. Choose an interval J containing y_o such that $m(J) < \frac{1}{k_o}$ and $\frac{m(E \cap J)}{m(J)} > \frac{1}{2}$. Since C_{k_o} is dense in J, there is a $t \in C_{k_o} \cap J$. Consequently, $\frac{m(T_t \cap J)}{m(J)} > \frac{1}{2}$, and it follows that $T_t \cap E \cap J$ is not empty. Let $s \in T_t \cap E \cap J$. Then $|f(s)| > \frac{2}{n_o}$ because $s \in E$. However, since $s \in T_t$, we have $|f(s) - f(t)| < \frac{1}{n_o}$, implying that $|f(s)| < \frac{2}{n_o}$. This contradiction completes our proof.

At this point it becomes natural to inquire whether or not for a locally integrable function the measure zero, first category set $\mathcal{A} \Delta \mathcal{L}$ must further be a σ-porous set . The next example shows that $\mathcal{A} \Delta \mathcal{L}$ need not be σ-porous even if f is bounded; indeed, $\mathcal{L} \setminus \mathcal{A}$ can contain any perfect set of measure zero. (The existence of perfect sets of measure zero which are not σ-porous, was first showed by Zajicek [9]; see also [5] by Humke and Thomson for additional methods for constructing such sets.)

EXAMPLE 2. Let P be any perfect set of measure zero. There is a bounded function $f: \mathbb{R} \to \mathbb{R}$ for which $P \subseteq \mathcal{L} \setminus \mathcal{A}$.

PROOF. We shall first define an auxiliary function g on [0, 1]. For each natural number n and for each $i = 0, 1, 2, \ldots, 2^n-1$, let

$$I_{n,i} = \left(\frac{1}{2^n} - \frac{i+1}{4^{n+1}}, \frac{1}{2^n} - \frac{i}{4^{n+1}}\right),$$

and define g initially on [0, 1/2] by letting $g(x) = (-1)^{n+i}$ for x in $I_{n,i}$, and $g(x) = 0$ otherwise. We then extend g to all of [0, 1] "symmetrically" about $\frac{1}{2}$ by letting $g(x) = -g(1 - x)$ for x in $[\frac{1}{2}, 1]$.

It is then an elementary exercise to show each of the following:

a) $m[g^{-1}(1)] = m[g^{-1}(-1)] = \frac{1}{2}$.

b) The right [left] density of each of the sets $g^{-1}(1)$ and $g^{-1}(-1)$ at zero [one] is $\frac{1}{2}$.

c) $\int_0^1 g(t)dt = 0$.

d) $\lim_{h\to0^+} \frac{1}{h}\int_0^h g(t)dt = \lim_{h\to0^+} \frac{1}{h}\int_0^h g(1-t)dt = 0$

Now let P be the given perfect set of measure zero. On each bounded open component, (a, b), of the complement of P we define f to be a copy of the function g; that is, for $x \in (a, b)$ we set $f(x) = g\left[\frac{x-a}{b-a}\right]$. Finally, for all other x we set $f(x) = 0$. It is then a straightforward matter to see that properties c) and d), together with the fact that P has measure zero, yield that every point in P is an L-point. (Indeed, the only non-L-points are those countably many in each bounded open component of the complement of P.) Furthermore, properties a) and b), together with the nullness of P, show that no point of P can be a point of approximate continuity of f, completing the proof. (For the purist, we should probably take note of the fact that it is possible to alter this example so that P exactly equals $\mathcal{L} \setminus \mathcal{A}$. This could be accomplished by replacing each of the jumps in the definition of g with a monotone differentiable interpolation on a small interval in such a way that the lengths of these interpolating intervals become sufficiently small as one approaches 0 and 1. Such an example would have $\mathbb{R} = \mathcal{L}$ and $\mathbb{R} \setminus \mathcal{A} = P$.)

We now turn our attention toward finding a condition on the "approximate behavior" of the function under which $\mathbb{R} \setminus \mathcal{L}$ will be σ-porous. Several things can be ruled out easily. Certainly, approximate differentiablity alone will not suffice, since it will not even guarantee the existence of a full measure set at each point of which f is integrable over a neighborhood. The following is an elementary example of such a function.

<u>EXAMPLE 3.</u> Let P be a nowhere dense, perfect set. There is an approximately differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ for which $\mathbb{R} \setminus \mathcal{L} = P$.

PROOF. Let $\{I_n = (a_n, b_n): n = 1, 2, \ldots\}$ denote the bounded open components of the complement of the nowhere dense, perfect set P. For each n let J_n denote the interval (c_n, d_n) whose midpoint is the same as the midpoint of I_n and whose length is $m(I_n)/2^n$. Then for each n define f on J_n in such a way that f is positive and differentiable on (c_n, d_n),

$$f(c_n) = f(d_n) = f'_+(c_n) = f'_-(d_n) = 0,$$

and

$$\int_{c_n}^{d_n} f(x)dx = 1.$$

Then let $f(x) = 0$ for all other $x \in \mathbb{R}$.

Clearly $\cup\{J_n: n = 1, 2, \ldots\}$ has density 0 at each point of P and, consequently, f is approximately differentiable everywhere. Since any neighborhood of a point $x \in P$ contains infinitely many of the J_n's, f fails to be integrable over such a neighborhood, and consequently, $x \in \mathbb{R} \setminus \mathcal{L}$. Points not in P are clearly in $\mathcal{L}$, completing the proof.

Similarly, approximate differentiability coupled with local integrability is not sufficient to force $\mathbb{R} \setminus \mathcal{L}$ to be σ-porous.

<u>EXAMPLE 4.</u> Let P be any perfect set of measure zero. There is a locally integrable, approximately differentiable function $f: \mathbb{R} \to \mathbb{R}$ for which $\mathbb{R} \setminus \mathcal{L} = P$.

PROOF. Let $\{I_n = (a_n, b_n): n = 1, 2, \ldots\}$ denote the bounded open components of the complement of the perfect measure zero set P. For each n let J_n denote the interval (c_n, d_n) whose midpoint is the same as the midpoint of I_n and whose length is $|I_n|/2^n$. Then for each n define f on J_n in such a way that f is positive and differentiable on (c_n, d_n),

$$f(c_n) = f(d_n) = f'_+(c_n) = f'_-(d_n) = 0,$$

and

$$\int_{c_n}^{d_n} f(x)dx = |I_n|.$$

Then let $f(x) = 0$ for all other $x \in \mathbb{R}$.

Clearly $\cup\{J_n: n = 1, 2, \ldots\}$ has density 0 at each point of P and, consequently, f is approximately differentiable everywhere. Let $x \in P$. Noting that if h is positive and $x + h = b_n$ for some n, then

$$\frac{1}{h}\int_0^h f(x + t)dt = 1,$$

the result readily follows.

Consequently, for locally integrable functions there does not seem to be any natural condition that one can place on the approximate behavior of the function to guarantee the σ-porosity of $\mathbb{R} \setminus \mathcal{L}$. The situation is, of course, much different for bounded functions, where we have $\mathcal{A} \subseteq \mathcal{L}$. For bounded functions, the assumption of approximate continuity would certainly do the job, but in an overkill type fashion. On the other hand, the assumption that f is bounded and of Baire class one is not sufficient to produce the σ-porosity of $\mathbb{R} \setminus \mathcal{L}$, as the characteristic function of a measure zero, non-σ-porous, perfect set shows. So, we look for some condition between those two extremes.

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be approximately symmetric at x if

$$\operatorname*{ap-lim}_{h \to 0} |f(x + h) + f(x - h) - 2f(x)| = 0.$$

Obviously, every approximately continuous function is approximately symmetric (approximately symmetric at every point) and Larson [6] has shown that approximately symmetric functions must belong to Baire class one.

THEOREM 2. Let $f: \mathbb{R} \to \mathbb{R}$ be bounded and approximately symmetric. Then $\mathbb{R} \setminus \mathcal{L}$ is a σ-porous set.

PROOF. Let K be a number such that $|f(x)| \le K$ for all real numbers x. Define the function $\phi: \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \int_0^x f(t)dt.$$

For each $h \in \mathbb{R}$ we have

$$\phi(x + h) - \phi(x - h) = \int_0^h (f(x + t) + f(x - t))dt.$$

Let x be an arbitrary real number and let E(x) denote a set having density one at 0 such that $\lim_{\substack{t \to 0 \\ t \in E(x)}} [f(x + t) + f(x - t) - 2f(x)] = 0$. Let ε be a positive number. Then there is a $\delta > 0$ such that $|f(x + t) + f(x - t) - 2f(x)| < \varepsilon$ for all $t \in (0, \delta) \cap E(x)$. Consequently, if $0 < h < \delta$, then

$$\left|\frac{\phi(x+h)-\phi(x-h)}{2h}-f(x)\right| = \left|\frac{1}{2h}\int_0^h [f(x+t)+f(x-t)-2f(x)]dt\right|$$

$$\leq \frac{1}{2h}\int_{E(x)\cap[0,h]} \varepsilon\, dt + \frac{1}{2h}\int_{[0,h]\setminus E(x)} 4K\, dt$$

$$\leq \frac{\varepsilon}{2} + \frac{2Km([0,\ h]\setminus E(x))}{h}.$$

Since $E(x)$ has density one at 0, this inequality implies that the symmetric derivative of ϕ exists at x and equals $f(x)$. Consequently, ϕ is symmetrically differentiable at each $x \in \mathbb{R}$ with symmetric derivative $f(x)$. According to Theorem 1 in [2], this implies that ϕ has $f(x)$ as an ordinary derivative at every point x except those in a σ-porous set. Clearly, any point at which $\phi'(x) = f(x)$ is an L-point of f.

The present authors find the above theorem interesting in light of the problems that we continue to have in trying to see if $\mathbb{R} \setminus \mathcal{A}$ must be σ-porous for a bounded approximately symmetric function.

We conclude this paper by noting that the exceptional set of the previous theorem need not be countable. Indeed, we shall show that $\mathbb{R} \setminus \mathcal{L}$ can be uncountable for a function in the more restricted class of L_p $(p \geq 1)$ smooth functions. A function $f: \mathbb{R} \to \mathbb{R}$ is said to be $\underline{L_p \text{ smooth}}$ at x if

$$\left[\frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2f(x)|^p dt\right]^{\frac{1}{p}} = o(h) \quad \text{as } h \to 0.$$

Then f is said to be L_p smooth if it is L_p smooth at each point in $\mathbb{R}$. It is known that every L_p smooth function is approximately symmetric and that the converse is false [8]. In [3] an L_p smooth function was constructed having uncountably many points of approximate discontinuity. Thus the next example is in some sense an improvement of that example.

EXAMPLE 5. For each $1 \leq p < \infty$ there is a bounded L_p smooth function $f: \mathbb{R} \to \mathbb{R}$ for which $\mathbb{R} \setminus \mathcal{L}$ is uncountable.

PROOF. It suffices to let p be a natural number. We shall first describe a certain Cantor set C in $I_0 = [0, 1]$ and then define our function, which will have the property that $C \subseteq \mathcal{L}$.

Let $L(0) = 1$, and for each natural number m, let

$$L(m) = \prod_{k=1}^{m} (k + 1)^{-(2p + 2)^{2(m - k + 1)}},$$

$$N(m) = \frac{L(m-1)}{L(m)},$$

$$D(m) = \left[\frac{1}{L(m)}\right]^{\frac{2p+1}{2p+2}},$$

and

$$M(m) = \frac{N(m) - D(m) - 2}{2}.$$

The set C is then defined as the intersection of a sequence of compact sets, I_n, where each I_n is the union of 2^n closed subintervals of I_{n-1}, each subinterval is of length $L(n)$, and each is $M(n)L(n)$ units from the nearest endpoint of the interval of I_{n-1} which contains it; more specifically, for each natural number n, let

$$I_n = \left\{ \sum_{m=1}^{\infty} k(m)L(m):\ k(m) = M(m) \text{ or } M(m)+D(m)+1 \text{ for } 1 \le m \le n, \right.$$
$$\left. \text{and } 0 \le k(m) \le N(m) \text{ for } m > n \right\},$$

and

$$C = \left\{ \sum_{m=1}^{\infty} k(m)L(m):\ k(m) = M(m) \text{ or } M(m)+D(m)+1 \right\}.$$

For each natural number n, we define f on $I_{n-1} \setminus I_n$ as follows. Let J be one of the 2^{n-1} component intervals of I_{n-1}. Partition J into $N(n)$ closed subintervals, each of length $L(n)$. On each of these closed subintervals of J, <u>except</u> the two in I_n, we let f assume the value 0 at each endpoint and midpoint, the value 1 on the open left half of the interval, and the value -1 on the open right half of the interval. This defines f on each $I_{n-1} \setminus I_n$, and hence on $[0, 1] \setminus C$. We let f assume the value 1 on $(1, \infty)$, -1 on $(-\infty, 0)$, and 0 on C. Thus f is defined on all of $\mathbb{R}$, and we now show that it has the desired properties.

We shall first show that f is L_p smooth. Clearly, f is smooth at each point of $\mathbb{R} \setminus C$; indeed, for each such x, there is a $\delta > 0$ such that for $0 < h < \delta$, $\Delta^2 f(x,h) = 0$, where $\Delta^2 f(x,h)$ denotes the expression $f(x + h) + f(x - h) - 2f(x)$. Now, for any $x \in C$ and $h > 0$, let

$$\psi_x(h) = \left[\frac{1}{h^{p+1}} \int_0^h |\Delta^2 f(x,t)|^p dt \right]^{\frac{1}{p}},$$

and

$$V_x(h) = \{t \in [0, h]: \Delta^2 f(x,t) \neq 0\}.$$

Then $m(V_x(h)) = m(\{t \in [0, h]: |\Delta^2 f(x,t)| = 2\})$ and

$$[\psi_x(h)]^p = \frac{2^p m(V_x(h))}{h^{p+1}}.$$

Consequently, to show that f is L_p smooth at a point $x \in C$, it will suffice to show that

$$m(V_x(h)) = o(h^{p+1}) \quad \text{as} \quad h \to 0.$$

For each $m \geq 2$ let $h_m = (N(m) - M(m))L(m)$. We shall show that for each $m \geq 2$, and each $x \in I_{m+1}$ for which $f(x) = 0$, we have

$$\frac{m(V_x(h))}{h^{p+1}} < \frac{4}{m} \quad \text{for} \quad h \in [h_m, h_{m-1}].$$

To this end, let $x \in I_{m+1}$ with $f(x) = 0$. In the following chart, we use $B_x(h)$ as an easily computed upper bound for $m(V_x(h))$, and we let $\gamma(m) = L(m) - 2L(m+1)M(m+1)$.

CHART 1

h	$B_x(h)$
h_{m+1}	h_{m+1}
$h_{m+1} + L(m)$	$h_{m+1} + \gamma(m)$
$h_{m+1} + D(m)L(m)$	$h_{m+1} + D(m)\gamma(m)$
$h_{m+1} + (D(m) + 1)L(m)$	$h_{m+1} + D(m)\gamma(m) + L(m)$
$M(m)L(m) + M(m+1)L(m+1)$	$h_{m+1} + L(m) + (M(m) - 2)\gamma(m)$
h_m	$h_{m+1} + L(m) + (M(m) - 2)\gamma(m) + {}$ $+ (D(m) + 2)L(m) - M(m+1)L(m+1)$

Before completing the chart, some explanation of the entries is in order at this point. (Recall that the only requirements on x is that it lie in I_{m+1} with $f(x) = 0$.) As progress toward establishing our claim for all $h \in [h_m, h_{m-1}]$, only the last entry in the above chart has any value. We have included the earlier entries, however, to help motivate the correctness of the final entry. By and large, we have been pessimistic in our bounds, $B_x(h)$. This is clearly the case in getting started as we set $B_x(h_{m+1}) = h_{m+1}$. Now consider the second line in the chart. When h is increased from h_{m+1} to $h_{m+1} + L(m)$, it is easily seen that the amount we must add to $B_x(h_{m+1})$ due to the additional length $L(m)$ will be the largest for that x which is located as far as it can be from the center of the interval $J = [a, b]$ of length $L(m)$ in I_m to which it belongs. Suppose, without loss of generality, that x belongs to the left one of the two intervals of length $L(m+1)$ in $I_{m+1} \cap J$. Then the worst situation occurs when $x = a + M(m+1)L(m+1)$. In this case, the amount we need to add to $B_x(h_m)$ is at worst the distance between $b + L(m)$ and the reflection of $a - L(m)$ about x. This distance is $b + L(m) - [2x - (a - L(m))] = \gamma(m)$.

Similarly, if we increased h by an additional length $L(m)$, then we would add another $\gamma(m)$ to obtain the corresponding $B_x(h_{m+1} + 2L(m)) = h_{m+1} + 2\gamma(m)$. This pattern continues as we add lengths of $L(m)$ to h_{m+1} a total of $D(m)$ times, which is how the entries in the third line of the chart are obtained.

As we increase h by yet another length $L(m)$ to bring us to the fourth line of the chart, we realize that $x + h$ (or $x - h$ in the situation where x were in the right member of the pair of intervals from I_{m+1} in J) can land in the other interval of I_{m+1} in J. We again take the pessimistic approach and increase our corresponding right hand chart entry by the full length L(m).

If at this point we were to increase h by a length L(m), we would again increase the corresponding right hand entry in the chart by $\gamma(m)$. This pattern would continue until we encountered the endpoint of the component interval of I_{m-1} to which x belongs. This produces the fifth line in the chart. Finally, the last line is obtained by again employing the pessimistic approach of treating all of the newly encountered h's as members of $V_x(h_m)$.

Before producing the rest of the chart to obtain $B_x(h_{m-1})$, let's pause here to verify that

$$\frac{B_x(h_m)}{h_m^{p+1}} < \frac{2}{m+1} .$$

We have

$$\begin{aligned} B_x(h_m) &= h_{m+1} + L(m) + (M(m)-2)\gamma(m) + (D(m)+2)L(m) - M(m+1)L(m+1) \\ &= [D(m)+M(m)+1]L(m) + [N(m+1)+2M(m+1)-2M(m+1)M(m)]L(m+1) \\ &< [D(m)+M(m)+1]L(m) + [2N(m+1)-2M(m+1)M(m)]L(m+1) \\ &= [D(m)+M(m)+1]L(m) +2L(m) - 2M(m+1)M(m)L(m+1) \\ &= D(m)L(m) + 3L(m) + M(m)L(m) - [N(m+1) - D(m+1) - 2]L(m+1)M(m) \\ &= D(m)L(m) + 3L(m) + [D(m+1) + 2]L(m+1)M(m) \\ &< 2D(m)L(m) + 2D(m+1)L(m+1)M(m). \end{aligned}$$

Next consider each term in this last sum divided by $(h_m)^{p+1}$. First,

$$\frac{2D(m)L(m)}{(h_m)^{p+1}} < \frac{2D(m)L(m)}{\left[\frac{N(m)L(m)}{2}\right]^{p+1}}$$

$$= \frac{2^{p+2}D(m)L(m)}{[L(m-1)]^{p+1}}$$

$$< \frac{2^{p+1}D(m)L(m)}{[L(m-1)]^{2p+2}}$$

$$= 2^{p+2}\left[\frac{L(m)}{L(m-1)^{(2p+2)^2}}\right]^{\frac{1}{2p+2}}$$

$$= \frac{2^{p+2}}{(m+1)^{2p+2}}$$

$$< \frac{1}{m+1}.$$

Likewise,

$$\frac{2D(m+1)L(m+1)M(m)}{(h_m)^{p+1}} < \frac{2D(m+1)L(m+1)M(m)}{[M(m)L(m)]^{p+1}}$$

$$< \frac{2D(m+1)L(m+1)}{[L(m)]^{p+1}}$$

$$< 2\left[\frac{L(m+1)}{L(m)^{(2p+2)^2}}\right]^{\frac{1}{2p+2}}$$

$$= \frac{2}{(m+2)^{2p+2}}$$

$$< \frac{1}{m+1}$$

Hence $\dfrac{B_x(h_m)}{h_m^{p+1}} < \dfrac{2}{m+1}$.

Now, returning to our charting procedure, and picking up where we left off at the end of Chart 1, we obtain:

CHART 2

h	$B_x(h)$
h_m	$B_x(h_m)$ (from Chart 1)
$h_m + L(m-1)$	$B_x(h_m) + \gamma(m-1)$
$h_m + D(m-1)L(m-1)$	$B_x(h_m) + \gamma(m-1)L(m-1)$
$h_m + (D(m-1) + 1)L(m-1)$	$B_x(h_m) + D(m-1)\gamma(m-1) + L(m-1)$
$M(m-1)L(m-1) + M(m)L(m)$	$B_x(h_m) + L(m-1) + (M(m-1) - 2)\gamma(m-1)$
h_{m-1}	$B_x(h_m) + L(m-1) + (M(m-1) - 2)\gamma(m-1) +$ $+ (D(m-1) + 2)L(m-1) - M(m)L(m)$

The rationale for the entries in this chart is exactly the same as that given for Chart 1, with the obvious exception that we are able to start with a better bound for $m(V_x(h_m))$, namely $B_x(h_m)$ from Chart 1, than we had for $m(V_x(h_{m+1}))$ at the start of Chart 1.

Based on the previous calculations, it is clear that

$$\frac{B_x(h_{m-1})}{h_{m-1}^{p+1}} < \frac{2}{m},$$

but we must show

$$\frac{m(V_x(h))}{h^{p+1}} < \frac{4}{m} \quad \text{for all } h \in [h_m, h_{m-1}],$$

whereas to this point we have established it only at the endpoints. However, it is not difficult to see from Chart 2 that the worst situation that need be considered is that arising in the fourth row. Let h_* denote $h_m + (D(m-1) + 1)L(m-1)$. We have

$$\begin{aligned} B_x(h_*) &= B_x(h_m) + D(m-1)\gamma(m-1) + L(m-1) \\ &= B_x(h_m) + (1 + D(m-1))L(m-1) - 2D(m-1)M(m)L(m) \\ &= B_x(h_m) + L(m-1) + D(m-1)[D(m) - 2]L(m) \\ &< B_x(h_m) + L(m-1) + D(m-1)D(m)L(m). \end{aligned}$$

Clearly, $B_x(h_m)/h_*^{p+1} < B_x(h_m)/h_m^{p+1}$, which we already know is less than 2/m. Next,

$$h_* = h_m + (D(m-1) + 1)L(m-1) > D(m-1)L(m-1),$$

and, consequently,

$$\begin{aligned} \frac{L(m-1)}{h_*^{p+1}} &< \frac{L(m-1)}{[D(m-1)L(m-1)]^{p+1}} \\ &= [L(m-1)]^{1/2} \\ &< \frac{1}{m}. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{D(m-1)D(m)L(m)}{h_*^{p+1}} &< \frac{D(m-1)D(m)L(m)}{[D(m-1)L(m-1)]^{p+1}} \\ &< \frac{D(m)L(m)}{L(m-1)^{p+1}} \\ &< \frac{D(m)L(m)}{L(m-1)^{2p+2}} \end{aligned}$$

$$= \left[\frac{L(m)}{L(m-1)^{(2p+2)^2}}\right]^{\frac{1}{2p+2}}$$

$$= \frac{1}{(m+1)^{(2p+2)}}$$

$$< \frac{1}{m} .$$

Consequently, we have that $B_x(h_*)/h_*^{p+1} < 4/m$, and hence the same inequality holds with h_* replaced by any $h \in [h_m, h_{m-1}]$.

From this, it immediately follows that f is L_p smooth at each $x \in C$. Consequently, f is L_p smooth at each $x \in \mathbb{R}$.

It now only remains to show that C contains no L-points for f. Let $x \in C$. For each n we have $x \in I_n$. Fix an n and let $J = [a, b]$ denote the component interval of I_n to which x belongs. Then J is of length $L(n)$. For $h = L(n)/2 + b - x$ we have

$$\frac{1}{h}\int_0^h f(x + t)\, dt = \frac{1}{h}\int_0^{b-x} f(x + t)dt + \frac{1}{h}\int_{b-x}^{b-x+L(n)/2} f(x + t)dt$$

$$= 0 + \frac{1}{h}\int_0^{L(n)/2} 1dt$$

$$= \frac{L(n)}{2h}$$

$$> \frac{1}{3} .$$

Since this inequality holds for a sequence of h's approaching zero, and since $f(x) = 0$, x cannot be an L-point for f, and our proof is complete.

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