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ON COMPOSITIONS WITH CONNECTED FUNCTIONS

Abstract: The main results are: Firstly, for any two surjections, f and g, of a real interval there exist connected surjections α and β such that $\alpha(f(x)) = g(\beta(x))$ for all x. Secondly, there exists a pair of connected functions whose composition is not connected, mod the continuum hypothesis.

Introduction. It is well known that a Darboux Baire 1 function on R can be "stretched" into a derivative or an approximately continuous function, in the sense that there exists a homeomorphism h such that f h is a derivative or approximately continuous (see [1], page 36). In general, one can ask what possible effects can a composition with a homeomorphism, on the inside or outside, have on a given type of function?

In this paper we initiate a study of this question when the homeomorphism restriction is relaxed to be just a surjection. Specifically we pose two general queries relative to two fixed classes $\mathcal{A}$ and $\mathcal{B}$ of surjections of a given open interval I.

Question 1 If $f, g \in \mathcal{A}$ do there exist $\alpha, \beta \in \mathcal{B}$ such that $\alpha \circ f = g \circ \beta$?

Question 2 If $f, g \in A$, do there exist $\alpha, \beta \in B$ such that $f = \alpha \circ g \circ \beta$?

In other words, with respect to the second question, given f and g can we "scramble" up both the domain and range of g (using functions in B) to produce f?

In general, given a specified class A of surjections we would like to find a more restrictive, yet interesting, class B for which the above equations have solutions.

In this paper we focus our attention mostly on taking B to be the family of all connected surjections of I, and we are able to obtain some interesting results as well as pose some interesting unsolved problems.

Throughout the sequel I will be an unspecified open interval. By c we mean $2^{\aleph_0}$. By $|A|$ is meant the cardinality of A. We say a set A is c-dense in I if each open subset of I contains c members of A. We will make no distinction between a function and its graph.

A function f from I into R is Darboux if it maps intervals onto intervals. A function f from I into R is connected if f is a connected subset of $I \times R$. We can characterize Darbouxness by the intermediate value property namely: for each a, b and λ the line segment $[a,b] \times \{\lambda\}$ hits f provided (a, f(a)) and (b, f(b)) lie on opposite sides. If we replace the "line segment" here by any continuum K with domain [a,b] and interpret "opposite" in terms of different components of $((\text{dom } K) \times R) - K$ we arrive at a characterization for connected functions (see [2]). This will be useful in the sequel.

Lemma 1 Let $f : I \to I$. If $|\text{rng } f| = c$, then there exists $A \subseteq I$ such that $f(I-A) \cap f(A) = \emptyset$ and both A and $I - f(A)$ are c-dense in I.

Proof: Let $\mathcal{G}$ consist of all those open intervals J such that $|f(J)| < c$. Put $G = \cup\mathcal{G}$. Clearly (1) $I - G \neq \emptyset$, since $|\text{rng } f| = c$; (2) $I - G$ is perfect; (3) $|f(G)| < c$; and (4) for any open subinterval J of I $J - G \neq \emptyset$ implies $|f(J)| = c$.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the family of all open subintervals of I which hit $I - G$ (resp. G). Let $\{z_\alpha\}_{\alpha<c}$ be a well-ordering of $\mathcal{A} \times c$ and $\{w_\alpha\}_{\alpha<c}$ be a well-ordering of $\mathcal{B} \times c$. For an ordered pair $\langle a,b \rangle$ define $F(\langle a,b \rangle) = a$.

By induction on c we choose $b_0 \in F(w_0) - f(G)$ and $a_0 \in F(z_0) - G$ and, in general, having defined a_ξ an b_ξ for each $\xi < \alpha$ we choose

$$b_\alpha \in F(w_\alpha) - f(G) - \{b_\xi : \xi < \alpha\} - \{f(a_\xi) : \xi < \alpha\}$$
$$a_\alpha \in F(z_\alpha) - G - \{a_\xi : \xi < \alpha\} - f^{-1}(\{b_\xi : \xi \leqq \alpha\}).$$

Clearly a_α and b_α exist for all $\alpha < c$. Put $B = \{b_\alpha : \alpha < c\}$ and $A' = G \cup \{a_\alpha : \alpha < c\}$.

For any non-void open subinterval H of I $|\{\alpha : F^{-1}(w_\alpha) = H\}| = c$. Therefore $|H \cap B| = c$ and B is c-dense in I. Likewise A' is c-dense in I.

Now suppose $B \cap f(A') \neq \emptyset$. Then since $B \cap f(G) = \emptyset$ there exists α and γ such that $b_\alpha = f(a_\gamma)$. Since $b_\alpha \notin \{f(a_\xi) : \xi < \alpha\}$ we must have $\alpha \leqq \gamma$. Since $a_\gamma \notin f^{-1}(\{b_\xi : \xi \leqq \gamma\})$ we must have $\gamma < \alpha$, a contradiction. Therefore, $B \cap f(A') = \emptyset$ and $I - f(A')$ is c-dense in I.

Finally put $A = f^{-1}(f(A'))$, then clearly A and $I - f(A)$ are c-dense in I and $f(A)$ and $f(I - A)$ are disjoint.

<u>Theorem</u> 1 <u>Let</u> $f,g : I \to I$. <u>If</u> $|\text{rng } f| = c$ <u>and</u> $g(I)$ <u>is an interval, there exist connected functions</u> α <u>and</u> β <u>taking on each value in</u> $g(I)$ <u>on each subinterval such that</u>

$$\alpha \circ f = g \circ \beta.$$

<u>In particular, if</u> f <u>and</u> g <u>are surjections of</u> I, <u>there exist connected surjections</u> α <u>and</u> β <u>such that</u> $\alpha \circ f = g \circ \beta$.

<u>Proof</u>: Let $\mathcal{C}$ consist of all closed sets in $I \times g(I)$ with domain a non-degenerate closed subinterval of I. Then if a function $G : I \to g(I)$ hits each member of $\mathcal{C}$ then G is connected and takes on each value in $g(I)$ over each subinterval. Let us omit well-order $\mathcal{C}$ as $\{C_\alpha : \alpha < c\}$.

By Lemma 1 choose A such that $f(I - A) \cap f(A) = \emptyset$ and both A and $I - f(A)$ are c-dense in I. Decompose A into c disjoint sets $\{A_\alpha : \alpha < c\}$ each c-dense in I. Decompose $I - f(A)$ into c disjoint sets $\{B_\alpha : \alpha < c\}$ each c-dense in I. Let $\{r_\alpha\}_{\alpha<c}$ be a well-ordering of $g(I)$. Pick $y_0 \in g(I)$.

Let $x \in A$. If $x \in A_\alpha \cap \text{dom } C_\alpha$ choose $h(x)$ so that $(x,h(x)) \in C_\alpha$. If $x \in A_\alpha - \text{dom } C_\alpha$ put $h(x) = y_0$. In each case define $k(f(x)) = g(h(x))$.

Let $y \in I - f(A)$. If $y \in B_\alpha \cap \text{dom } C_\alpha$ choose $k(y)$ so that $(y, k(y)) \in C_\alpha$. If $y \in B_\alpha - \text{dom } C_\alpha$ put $k(y) = y_0$.

If $x \notin A$, then $f(x) \in B_\alpha$ for some α since $F(I - A) \cap f(A) = \emptyset$.

Since $K(I) \subseteq g(I)$ we may choose $h(x) \in g^{-1}(K(f(x)))$.

Obviously $k \circ f = g \circ h$. Also clearly each C_α hits g and k so g and k are connected and take on each value in $g(I)$ on each subinterval.

Corollary 1 *If f is any surjection, then there exist connected surjections α and β such that $\alpha \circ f$ and $f \circ \beta$ are connected.*

We can obtain the following variant of the above result.

Theorem 2 *If f is any surjection, then there exists a measurable, Darboux surjection β such that $f \circ \beta$ is measurable and Darboux.*

Proof: Let $\{V_n\}_{n=1}^{\infty}$ be an open basis for I. Choose sequences of non-void nowhere dense null perfect sets $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ such that $A_n \subseteq V_n$, $B_n \subseteq V_n$ and $A \cap B = \emptyset$ where $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$.

We can find a Baire 2 function h on A such that $h(A_n) = I$ for each n. Then define $k(x) = f(h(x))$ for each $x \in A$. Likewise we can find a Baire 2 function k on B such that $k(B_n) = I$ for each n. For $x \in B$ select $h(x) \in f^{-1}(k(x))$. For $x \in I - A - B$ define $h(x) = 0$ and $k(x) = f(0)$. (Assume I contains 0)

Clearly $k = f \circ h$ and h and k being constant except on the null set $A \cup B$ must be measurable. Moreover, h and k are Darboux because they map each subinterval onto I.

Now we turn our attention to addressing Question 2. We will find that Theorem 1 has no direct analogue. Let us say that a surjection g *can be scrambled via functions in* a class C *into* f if $f = \alpha \circ g \circ \beta$ has solutions in C. Then, we have the following characterization of scrambling.

Theorem 3 *A surjection* g *can be scrambled into a surjection* f *via surjections if and only if there exists a decomposition of* I, $\{A(y) : y \in I\}$, *into disjoint non-empty sets such that for all* $y \in I$

$$|\cup\{g^{-1}(z) : z \in A(y)\}| \leqq |f^{-1}(y)|.$$

Moreover, g *can be scrambled into* f *via permutations if and only if there exists a permutation* p of I *such that for each* $y \in I$

$$|g^{-1}(y)| = |f^{-1}(p(y))|.$$

Proof: Suppose $f = \alpha \circ g \circ \beta$. Define $A(y) = \alpha^{-1}(y)$. Then $\beta(f^{-1}(y)) = g^{-1}(\alpha^{-1}(y)) = \cup\{g^{-1}(z) : z \in A(y)\}$. Since $|\beta(f^{-1}(y))| \leqq |f^{-1}(y)|$ the conditions holds.

On the other hand suppose the condition holds. Define α by $\alpha(x) = y$ whenever $x \in A(y)$. Define β on each $f^{-1}(y)$ so that $\beta(f^{-1}(y)) = \cup\{g^{-1}(z) : z \in A(y)\}$. Clearly $f = \alpha \circ g \circ \beta$.

The additional assertion for permutation solutions follows similarly.

Theorem 4 *If a surjection* f *has all its level sets of cardinality* c, *then each surjection can be scrambled into* f. *In particular, there is a continuous function* g *such that each surjection can be scrambled* *via surjections into* g. *The identity function can be scrambled into any surjection*.

Proof: For such an f the criterion of Theorem 2 is easily established. The example of Foran of a continuous nowhere-differentiable function g from [0,1] onto [0,1] has all its level sets nonempty perfect sets (see page 223 [1]). It is easy to construct from this a continuous g from I onto I having all its level sets uncountable. For the last assertion apply Theorem 2 where $A(y) = \{y\}$.

Any two surjections are not necessarily comparable by scrambling. For example, take f to be any continuous function having each level set countably infinite. Pick g to be any continuous function having one level set uncountable and all others finite. Then according to Theorem 3 neither of these functions can be scrambled into the other.

In light of Theorem 2 Question 2 would have to be reduced to: *if* $f = \alpha \circ g \circ \beta$, *can* α *and* β *be selected to be connected surjections*? The answer is *no* even for Darboux surjections because taking g to be the identity function and f to be any non-Darboux function we would have a composition of two Darboux functions not being Darboux. This is a contradiction since Darbouxness is preserved by composition.

The foregoing also suggests the following question: *is every Darboux function the composition of two connected functions*? Or in the light of the next theorem, *is* f *Darboux iff* f *is the composition of connected functions*? This problem seems exceedingly difficult to answer.

The set-theoretic assumption needed in the next result is also a consequence of the continuum hypothesis or Martin's Axiom.

Theorem 5 Connectedness is not preserved under compositions, provided the union of less than $2^{\aleph_0}$ nowhere dense sets is meager.

Proof: Let $I = [0,1]$. Let $\{A_\alpha : \alpha < c\}$ be a decomposition of I into disjoint countable sets each dense in I. Let $\{r_\alpha : \alpha < c\}$ be a well-ordering of I, where $r_0 \neq 0$. Define for $x \in A_\alpha$

$$f(x) = \begin{cases} r_\alpha & \text{if} \quad x \neq r_\alpha \\ \frac{1}{2} r_\alpha & \text{if} \quad x = r_\alpha \end{cases}$$

Then $f : I \to I$ fails to intersect the diagonal yet each level set of f is countable and dense in I. In particular, f is Darboux but not connected.

We will show that f is a composition of two connected functions. Let K be the set of all continua in $I \times I$ with an interval as a domain. By a result in [2] any function hitting all members of K will be connected. Let E and F denote the even and odd ordinals respectively less than c. Let K be well-ordered by $\{E_\alpha : \alpha \in E\}$ and also by $\{F_\alpha : \alpha \in F\}$ such that E_0 and F_1 are both $I \times \{0\}$.

By induction we will construct functions h_α and g_α for $\alpha < c$ as follows:

Let $\{s_n\}_{n=0}^{\infty}$ be a countably dense sequence in I with $s_0 = 0$. Decompose $f^{-1}(0)$ into countably many disjoint sets $\{B_n\}_{n=0}^{\infty}$ each of which is dense in I. Define $g_0(x) = s_n$ if $x \in B_n$ and $h_0(s_n) = 0$. Put $g_1 = g_0$ and $h_0 = h_1$. Then, g_0 hits E_0 and h_1 hits F_1. Moreover, $h_1 \circ g_1 = f|(f^{-1}(0))$.

Now suppose for each $\alpha < \beta$ we have constructed functions g_α and h_α such that

(1) $g_\alpha \subseteq g_\gamma$ and $h_\alpha \subseteq h_\gamma$ when $\alpha < \gamma$

(2) $|\text{dom } h_\alpha| \leqq \aleph_0 \cdot |\alpha + 1|$, $|\text{dom } g_\alpha| \leqq \aleph_0 \cdot |\alpha + 1|$

(3) g_α hits E_α when α is even and h_α hits F_α when α is odd

(4) $h_\alpha \circ g_\alpha = f|(\text{dom } g_\alpha)$.

Suppose β is even. If E_β hits $\cup\{g_\alpha : \alpha < \beta\}$ at a point of some g_γ then define $g_\beta = g_\gamma$ and $h_\beta = h_\gamma$. If E_β misses $\cup\{g_\alpha : \alpha < \beta\}$, then for each $\lambda \in \text{dom} \cup\{h_\alpha : \alpha < \beta\}$, $(I \times \{\lambda\}) \cap E_\beta$ is nowhere dense in $I \times \{\lambda\}$. Since $|\text{dom} \cup\{h_\alpha : \alpha < \beta\}| \leqq \Sigma \{|\text{dom } h_\alpha| : \alpha < \beta\} \leqq |\beta| \, |\alpha + 1| \cdot \aleph_0 < c$ we may apply the set theoretic assumption to conclude that the set $\Gamma = \text{dom}(E_\beta \cap \cup \{I \times \{\lambda\} : \lambda \in \text{dom} \cup\{h_\alpha : \alpha < \beta\}\})$ is meager in $\text{dom } E_\beta$. Also since $|\text{dom} \cup\{g_\alpha : \alpha < \beta\}| < c$, $\text{dom}(E_\beta \cap \cup\{g_\alpha : \alpha < \beta\})$ is also meager in $\text{dom } E_\beta$. Since, for $k \in K$, dom k is a non-degenerate interval it is not meager so we can find $a \in \text{dom } E_\beta - \text{dom} \cup \{g_\alpha : \alpha < \beta\}$ and $a, b \notin \text{dom} \cup \{h_\alpha : \alpha < \beta\}$ such that $(a,b) \in E_\beta$.

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Let $\lambda = f(a)$ and decompose $f^{-1}(\lambda)$ into countably many disjoint sets $\{\beta_n\}_{n=0}^{\infty}$ each dense in I. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a dense sequence in $I - \text{dom} \cup \{h_\alpha : \alpha < \beta\}$ where $\lambda_o = b$. Put

$$g_\beta(x) = \begin{cases} g_\alpha(x) & \text{if } x \in \text{dom } g_\alpha \\ \lambda_n & \text{if } x \in A_n \end{cases}$$

and put

$$h_\beta(y) = \begin{cases} h_\alpha(y) & \text{if } y \in \text{dom } h_\alpha \\ \lambda_0 & \text{if } y = \lambda_n. \end{cases}$$

Now suppose β is odd. If F_β hits $\cup\{h_\alpha : \alpha < \beta\}$ at a point of some h_γ put $h_\beta = h_\gamma$ and $g_\beta = g_\gamma$. If F_β misses $\cup\{h_\alpha : \alpha < \beta\}$, then using the same argument in the case where β is even there exists $(a,b) \in F_\beta$ such that $b \notin \text{rng } \{h_\alpha : \alpha < \beta\}$ and $a \notin \text{dom} \cup \{h_\alpha : \alpha < \beta\}$. Let $\{s_n\}_{n=0}^{\infty}$ be a sequence in $I - \text{dom} \cup \{h_\alpha : \alpha < \beta\}$ such that $s_0 = a$. Decompose $f^{-1}(b)$ into countably many disjoint sets $\{B_n\}_{n=0}^{\infty}$ each dense in I. Define

$$g_\beta(x) = \begin{cases} g_\alpha(x) & \text{if } x \in \text{dom } g_\alpha \\ s_n & \text{if } x \in \beta_n \end{cases}$$

and

$$h_\beta(y) = \begin{cases} h_\alpha(y) & \text{if } x \in \text{dom } h_\alpha \\ b & \text{if } y = s_n \end{cases}$$

It is easily checked that the inductive hypotheses (1) through (4) are satisfied.

Now define $g = \cup\{g_\alpha : \alpha < c\}$ and $h = \cup\{h_\alpha : \alpha < c\}$. Since each member of K is some E_ξ and some F_μ it follows that $\text{dom } h = \text{dom } g = \text{rng } h = \text{rng } g = I$. Then (1) and (4) imply that $f = h \circ g$. Moreover, by (3) both g and h are connected.

Bibliography

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