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RINGS OF BAIRE FUNCTIONS ON REALCOMPACT SPACES

1. Introduction. Let $C(X)$ denote the ring of continuous real valued functions on a completely regular Hausdorff space X, let $E(X)$ denote the ring of Baire functions on X, and let $D(X)$ denote the ring of Baire class one functions on X. (Any function in $D(X)$ is the pointwise limit of a sequence of functions in $C(X)$.) Then $C(X) \subset D(X) \subset E(X)$. We let $C^+(X)$ (respectively $D^+(X)$, $E^+(X)$) denote the set of nonnegative functions in $C(X)$ (respectively $D(X)$, $E(X)$). By a nonnegative linear functional (written nlf) F on $C(X)$ or $D(X)$ or $E(X)$, we mean a real valued linear function that takes nonnegative functions in its domain to nonnegative numbers and takes the function 1 to the number 1. (In our notation, functions in $E(X)$ will be lower case, the identity in the ring $E(X)$ is denoted 1, and mappings defined on $E(X)$, $D(X)$ and $C(X)$ are upper case. If $f, g \in E(X)$, then $f \vee g$ denotes the maximum of f and g, and $f \wedge g$ denotes the minimum of f and g.)

The sets $C(X)$, $D(X)$ and $E(X)$ have some different properties. For example, $E(X)$ is closed under pointwise convergence of sequences, but $D(X)$ and $C(X)$ are not in general. On the other hand, they are all closed under uniform convergence ([2], p. 138). $E(X)$ contains the characteristic function of any Baire set, but $C(X)$ and $D(X)$ do not in general. $D(X)$ and $E(X)$ contain the characteristic function of any zero-set ([1], pp. 14-15), but $C(X)$ does not in general. (Note that if $f \in C(X)$, then $\lim_{n\to\infty} (1-(|f|\wedge 1))^n$ lies in $D(X)$ and $E(X)$.)

We are particularly interested in spaces that are realcompact [1]. The following result is well-known (see [3], Theorems 17 and 18).

Theorem 1. Let F be an nlf on $C(X)$ where X is realcompact. Then there exists a compact subset Y of X such that any $f \in C(X)$ with $f(Y) = 0$ satisfies $F(f) = 0$, and such that any $f \in C^+(X)$ with $f(Y) \neq 0$, satisfies $F(f) > 0$. Moreover, there is a Baire measure m on X such that for all $g \in C(X)$,

$$\int_X g \, dm = F(g).$$

We will give a different proof of Theorem 1 and prove analogues for $D(X)$ and $E(X)$ in which Y must be a finite set. We will find that every nlf on $D(X)$ or $E(X)$ is continuous in the topology of pointwise convergence, any nlf on $D(X)$ extends to an nlf on $E(X)$, and an nlf on $C(X)$ extends to an nlf on $E(X)$ if and only if it is continuous in the topology of pointwise convergence (Theorems 3 and 4).

Just as any ring isomorphism of $C(X_1)$ onto $C(X_2)$ (X_i realcompact) is implemented by a homeomorphism of X_2 onto X_1, a ring isomorphism of $D(X_1)$ onto $D(X_2)$ or $E(X_1)$ onto $E(X_2)$ is implemented by a bijective mapping of X_2 onto X_1. In the $E(X_i)$ case, under the bijection, Baire sets correspond to Baire sets, and in the $D(X_i)$ case, Baire class one F_σ-sets correspond to Baire class one F_σ-sets. (By a Baire class one F_σ-set we mean the union of countably many zero-sets. By a Baire class one G_δ-set we mean the complement of a Baire class one F_σ-set.) This is contained in Theorem 6.

We use the topology of pointwise convergence to determine when a ring homomorphism of $C(X_1)$ to $C(X_2)$, or $D(X_1)$ to $D(X_2)$, or $E(X_1)$ to $E(X_2)$ is an isomorphism onto the second ring (X_i realcompact). The necessary and sufficient condition is that closed subsets of the first ring map to closed subsets of the second ring (Theorem 7).

We do not determine if every metrizable space is realcompact. But we do provide another necessary and sufficient condition for a metrizable space to be realcompact (Theorem 11).

2. <u>Realcompact spaces</u>. Let F be a nonnegative linear functional on $C(X)$ where X is a realcompact space. Embed X in its Stone-Čech

compactification S. Any function $f \in C^+(X)$ extends to a unique continuous nonnegative extended real valued function on S; to save notation, call the extension f also.

Now let $f \in C^+(X)$ such that $F(f) = 0$. Then $A = f^{-1}(0) \subset S$ is nonvoid; for otherwise f is positive and bounded away from 0 on the compact space S, and for some positive number c, $f \geq c1$, and

$$F(f) \geq F(c1) = cF(f) = c > 0.$$

Let $\mathfrak{a}$ be the family of all compact subsets of S of the form $f^{-1}(0)$ where $f \in C^+(X)$ and $F(f) = 0$. For example $S \in \mathfrak{a}$. Then every set in $\mathfrak{a}$ is nonvoid, and $\mathfrak{a}$ is closed under finite intersections. (The second statement follows from

$$F(f_1+f_2) = F(f_1) + F(f_2) = 0 \quad \text{and} \quad (f_1+f_2)^{-1}(0) = f_1^{-1}(0) \cap f_2^{-1}(0).)$$

Then $Y = \cap \mathfrak{a}$ is a nonvoid compact subset of S. Moreover, if $g \in C^+(X)$ and g is positive at some point in Y, then $F(g) > 0$. We make the following observations about Y.

1. $Y \subset X$.

<u>**Proof.**</u> Suppose, to the contrary, $y \in Y \setminus X$. Since X is realcompact, there is an $f \in C^+(X)$ such that $f(y) = \infty$. For each positive integer n, put $g_n = (f \vee n) - n1$. Then $g_n \in C^+(X)$ and $g_n(y) = \infty$ for all n.

Moreover, $F(g_n) > 0$ and $g = \sum_{n=1}^{\infty} g_n/F(g_n) \in C^+(X)$. (Note that if $x \in X$ and $f(x) < k$, then g_n vanishes on the nbhd. $f^{-1}[0,k)$ of x for $n > k$.) Finally, $g \geq \sum_{n=1}^{N} g_n/F(g_n)$ and

$$F(g) \geq F(\sum_{n=1}^{N} g_n/F(g_n)) = N$$

for any integer $N > 0$. But this is impossible, and assertion 1 is proved. □

__Definition.__ We call each point in Y a __heavy__ point of F. All other points in X we call __light__ points of F.

2. If $g \in C^+(X)$ and $g^{-1}(0,\infty)$ is separated from Y, then $F(g) = 0$.

__Proof.__ Suppose, to the contrary, that $F(g) > 0$. For each $f \in C(X)$, define $F_0(f) = F(fg)/F(g)$. Then clearly F_0 is a nonnegative linear functional on $C(X)$. By assertion 1, F_0 has a heavy point $w \in X$.

Let $h \in C^+(X)$ such that $h(w) > 0$. Let $h_0 \in C^+(X)$ such that $h_0(w) > 0$, $h_0 \leq h$, and g is bounded on the set $h_0^{-1}(0,\infty)$. Say $g < c$ on this set where c is constant. Then

$$0 < F_0(h_0) = F(h_0 g)/F(g) \leq F(ch_0)/F(g) = cF(h_0)/F(g)$$

and $0 < F(h_0) \leq F(h)$. It follows that w is also a heavy point of F and $w \in Y$. But $g^{-1}(0,\infty)$ is separated from Y, so there is an $f \in C^+(X)$ with $f(w) > 0$ and $fg = 0$. So $F_0(f) = F(fg)/F(g) = 0$ and w is a light point of F_0. This contradiction proves assertion 2. □

3. If $g \in C(X)$ and g vanishes on Y, then $F(g) = 0$.

__Proof.__ It suffices to let $g \in C^+(X)$ because $g = (g \vee 0) + (g \wedge 0)$. Take any $\varepsilon > 0$. Then $((g \vee \varepsilon) - \varepsilon 1)^{-1}(0,\infty)$ is separated from Y and by assertion 2,

$$0 = F((g \vee \varepsilon) - \varepsilon 1) = F(g \vee \varepsilon) - \varepsilon \geq F(g) - \varepsilon.$$

Thus $F(g) \leq \varepsilon$, and because ε is arbitrary, $F(g) = 0$. This proves assertion 3. □

4. Thus if $f_1 = f_2$ on Y, and $f_1, f_2 \in C(X)$, we have $F(f_1) = F(f_2)$. Moreover, Y is a compact and closed subset of S, and by the Tietze extension theorem, any function in $C(Y)$ extends to bounded functions in $C(S)$ and $C(X)$. It follows that any zero-set with respect to Y is the intersection of Y with a zero-set with respect to X. Hence any Baire set with respect to Y is the intersection of Y with a Baire set with respect to X. Conversely, any such intersection with Y is a Baire set with respect to Y.

Now F defines an obvious nonnegative linear functional F_1 on C(Y). To wit,

$$F(h) = F_1(h_1)$$

where h_1 is the restriction of h to Y. By [4], there is a Baire measure m_1 on Y such that $F_1(h_1) = \int_Y h_1 \, dm_1$ for $h_1 \in C(Y)$. But m_1 defines an obvious Baire measure m on X. To wit, $m(B) = m_1(B \cap Y)$ for all Baire sets B in X. It follows from the definition of the integral that

$$F(h) = F_1(h_1) = \int_Y h_1 \, dm_1 = \int_X h \, dm,$$

for all $h \in C(X)$. □

This discussion provides an alternative proof to Theorem 1. Compare with Theorems 17 and 18 of [3].

We next observe that for nlfs F on C(X), D(X) or E(X), functions in the domain of F behave as if they are truncated.

__Lemma 1__. Let F be an nlf on C(X) (respectively, D(X), E(X)) and let f lie in the domain of F. Then there is a positive number c such that

$$F((f \vee c) - c1) = F((f \wedge (-c)) + c1) = 0 \quad \text{and} \quad F((f \wedge c) \vee (-c)) = F(f).$$

__Proof__. Let $r_n = \tan(\frac{1}{2}\pi - 2^{-n})$. Then $r_n \uparrow \infty$. Since $f = (f \vee 0) + (f \wedge 0)$, it suffices to let f be nonnegative. Suppose that $F((f \vee c) - c1) > 0$ for all numbers c. For integers $n > 0$, put $f_n = (f \vee r_n) - r_n 1$. Put $h = f + \sum_{n=1}^{\infty} f_n/F(f_n)$. It follows that h is finite on X. Moreover, $h \in C(X)$ if $f \in C(X)$, and $h \in E(X)$ if $f \in E(X)$. Thus in cases C(X) and E(X) we have $h \geq \sum_{n=1}^{N} f_n/F(f_n)$ and $F(h) \geq F(\sum_{n=1}^{N} f_n/F(f_n)) = N$ for all N, which is impossible. This proves Lemma 1 for $C(X)$ and $E(X)$.

Likewise, to prove Lemma 1 for $D(X)$, it suffices to prove that $h \in D(X)$ if $f \in D(X)$. For each positive integer N put

$$g_N = \arc\tan\,(f + \sum_{n=1}^{N} f_n/F(f_n)).$$

Then $\arc\tan h = \lim_{N\to\infty} g_N$ pointwise on X. At each $x \in X$ where $g_N(x) > g_{N-1}(x)$ and $N > 1$, we have $f_N(x) > 0$, $f(x) > r_N$, $\arc\tan f(x) > \frac{1}{2}\pi - 2^{-N}$, $\frac{1}{2}\pi > g_N(x) > g_{N-1}(x) \geq \frac{1}{2}\pi - 2^{-N}$, and $g_N(x) - g_{N-1}(x) < 2^{-N}$. So $|g_N - g_{N-1}| \leq 2^{-N}$. Each $g_N \in D(X)$, so let $p_{Nj} \in C(X)$ such that $0 \leq p_{Nj} \leq 2^{-N}$ and $\lim_{j\to\infty} p_{Nj} = g_N - g_{N-1}$ pointwise on X for each N.

Put $q_n = p_{1,n-1} + p_{2,n-2} + p_{3,n-3} + \cdots + p_{n-1,1}$ for each integer $n > 1$. For $t \in X$,

$$g_N(t) - g_1(t) + 2^{1-N} = \sum_{n=2}^{N} (g_n - g_{n-1})(t) + 2^{1-N} \geq \limsup_{n\to\infty} q_n(t),$$

$$g_N(t) - g_1(t) - 2^{1-N} = \sum_{n=2}^{N} (g_n - g_{n-1})(t) - 2^{1-N} \leq \liminf_{n\to\infty} q_n(t).$$

It follows that $-g_1(t) + \arc\tan h(t) = \lim_{n\to\infty} q_n(t)$ pointwise on X, and since each q_n is continuous, $-g_1 + \arc\tan h \in D(X)$. But $\tan g_1 \in D(X)$, and hence $g_1 \in D(X)$. Finally $\arc\tan h \in D(X)$; say $\lim h_n = \arc\tan h$ pointwise on X where each $h_n \in C^+(X)$. So $\lim_{n\to\infty} \tan((\frac{1}{2}\pi - n^{-1}) \wedge h_n) = h$ pointwise on X and $h \in D(X)$. □

Next we show that if F is an nlf on $D(X)$ or $E(X)$, then F has at most a finite number of heavy points. (Recall that x is a heavy point of F if for each $f \in C^+(X)$ with $f(x) > 0$, we have $F(f) > 0$.) This is true for completely regular Hausdorff spaces, realcompact or not.

<u>Lemma 2</u>. Let F be an nlf on $D(X)$ or $E(X)$ where X is a completely regular Hausdorff space. Then F has at most finitely many heavy points.

Proof. Suppose, to the contrary, that there are infinitely many heavy points of F. Let x_1 be a heavy point such that some nbhd. of x_1 excludes infinitely many other heavy points. (If a heavy point has no such nbhd. then any other heavy point will have one.) Let U_1 be an open nbhd. of x_1 such that $X \setminus \bar{U}_1$ contains infinitely many heavy points. Likewise, choose a heavy point x_2 and an open nbhd. U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$ and $X \setminus (\bar{U}_1 \cup \bar{U}_2)$ contains infinitely many heavy points. We use induction on n to choose a heavy point x_n and an open nbhd. U_n of x_n such that $U_i \cap U_j = \emptyset$ if $i \neq j$ and infinitely many heavy points of F lie in $X \setminus (\bar{U}_1 \cup \cdots \cup \bar{U}_n)$. Now let $f_n \in C^+(X)$ such that $f_n(x_n) > 0$ and f_n vanishes outside U_n for each n. Put $f = \sum_{n=1}^{\infty} f_n/F(f_n)$. Then $f \in D(X) \subset E(X)$ and for each N,

$$f \geq \sum_{n=1}^{N} f_n/F(f_n) \quad \text{and hence} \quad F(f) \geq N$$

which is impossible. □

The next Lemma on uniqueness of measures is not very original, but we will need it. It does not require realcommpactness.

Lemma 3. Let m_1 and m_2 be Baire measures on a space X such that $m_1(X) = m_2(X) = 1$ and for each bounded function $f \in C(X)$,

$$\int f \, dm_1 = \int f \, dm_2.$$

Then $m_1 = m_2$.

Proof. Consider the family $\mathcal{F}$ of all functions $f \in E(X)$ for which

$$\int ((1 \wedge f) \vee 0) \, dm_1 = \int ((1 \wedge f) \vee 0) \, dm_2.$$

Clearly $C(X) \subset \mathcal{F}$. Moreover, if $g \in E(X)$ is the pointwise limit of a sequence of functions $(g_n) \subset \mathcal{F}$, then for each n,

$$\int ((1 \wedge g_n) \vee 0) \, dm_1 = \int ((1 \wedge g_n) \vee 0) \, dm_2$$

and by the Lebesgue dominated convergence theorem,

$$\int((1\wedge g) \vee 0)\, dm_1 = \int ((1\wedge g) \vee 0)\, dm_2.$$

Then $\mathcal{F}$ is closed under pointwise limits of sequences, and hence $\mathcal{F} = E(X)$.

Now let g be the characteristic function of any Baire set A. Then $g \in E(X)$ and

$$m_1(A) = \int g\, dm_1 = \int g\, dm_2 = m_2(A). \qquad \square$$

We show that any nlf on $E(X)$ is an integral. Again realcompactness is not needed here.

<u>Lemma 4</u>. Let F be an nlf on $E(X)$. Then there is a unique Baire measure m on X such that for all $f \in E(X)$,

$$\int f\, dm = F(f).$$

<u>Proof</u>. For each Baire set A, put $m(A) = F(k_A)$ where k_A is the characteristic function of A. Then $m(\emptyset) = 0$, $m(X) = 1$, and m is obviously finitely additive. To prove that m is countably additive let $A_1, A_2, A_3, \ldots$ be a sequence of mutually disjoint Baire sets and suppose, to the contrary, that

$$\sum_{j=1}^{\infty} m(A_j) \neq m(\bigcup_{j=1}^{\infty} A_j).$$

For each N,

$$\sum_{j=1}^{N} m(A_j) \neq m(\bigcup_{j=1}^{N} A_j) \leq m(\bigcup_{j=1}^{\infty} A_j),$$

so $\sum_{j=1}^{\infty} m(A_j) < m(\bigcup_{j=1}^{\infty} A_j)$.

For each N, let $B_N = (\bigcup_{j=N}^{\infty} A_j)$. Then $m(B_N) > 0$ for all N, and $\bigcap_{N=1}^{\infty} B_N = \emptyset$. Finally

$$g = \sum_{N=1}^{\infty} k_{B_N} / m(B_N) \in E(X),$$

and for all $j > 0$,

$$F(g) \geq F(\sum_{N=1}^{j} k_{B_N} / m(B_N)) = j,$$

which is impossible. Thus m is a Baire measure on X.

If $f \in E(X)$, then $f = (f \vee 0) + (f \wedge 0)$. Thus it suffices to take $f \in E^+(X)$.

Fix $f \in E^+(X)$, and choose any $\varepsilon > 0$. Then there is an index $N > 0$ such that $F(f \wedge (N\varepsilon)) = F(f)$ by Lemma 1. Now let $B_j = p^{-1}[(j-1)\varepsilon, j\varepsilon)$ where $p = f \wedge (N\varepsilon)$. We note that

$$f \wedge (N\varepsilon) \leq \sum_{j=1}^{N+1} j\varepsilon k_{B_j} \leq \sum_{j=1}^{N+1} (j-1)\varepsilon k_{B_j} + \varepsilon 1$$

and

$$F(f) = F(f \wedge (N\varepsilon)) \leq \sum_{j=1}^{N+1} (j-1)\varepsilon F(k_{B_j}) + \varepsilon = \sum_{j=1}^{N+1} (j-1)\varepsilon m(B_j) + \varepsilon \leq \int f \, dm + \varepsilon.$$

Also

$$f \geq \sum_{j=1}^{\infty} (j-1)\varepsilon k_{B_j} \geq \sum_{j=1}^{\infty} j\varepsilon k_{B_j} - \varepsilon 1$$

and

$$F(f) \geq \sum_{j=1}^{\infty} j\varepsilon F(k_{B_j}) - \varepsilon = \sum_{j=1}^{\infty} j\varepsilon m(B_j) - \varepsilon \geq \int f \, dm - \varepsilon.$$

Since ε is arbitrary, $F(f) = \int f \, dm$ for $f \in E^+(X)$, and clearly for $f \in E(X)$. Uniqueness of the measure m follows from Lemma 3. □

The next Lemma concerns only $D(X)$.

<u>Lemma 5</u>. Let $f \in C(X)$, $0 \leq f \leq 1$ and let $A = f^{-1}(1)$. Then

$$\sum_{n=1}^{\infty} (f^n)\cdot(1 - k_A) \in D(X)$$

where k_A is the characteristic function of A.

<u>Proof</u>. Let $g = \sum_{n=1}^{\infty} (f^n)\cdot(1 - k_A)$. If $x \in X$ and $0 \leq f(x) < 1$, then $\sum_{n=1}^{\infty} f(x)^n < \infty$ and $g(x)$ is real. If $x \in X$ and $f(x) = 1$, then $1 - k_A(x) = 0$ and $g(x) = 0$. Thus $g < \infty$ on X. Also

$$g(x) = \lim_{N\to\infty} (1 - k_A(x)) \sum_{n=1}^{N} f(x)^n$$

for each $x \in X$, and $f(x) = 1$ if $k_A(x) = 1$. It follows that

$$g(x) = \lim_{N\to\infty} (1 - f(x)^N) \sum_{n=1}^{N} f(x)^n$$

and since each term on the right lies in $C(X)$, $g \in D(X)$. □

We are now able to analyze the situation when F has exactly one heavy point. We assume realcompactness.

<u>Lemma 6</u>. Let F be an nlf on $C(X)$ or $D(X)$ or $E(X)$ where X is realcompact. Suppose F has exactly one heavy point x_0. Then $F(f) = f(x_0)$ for any f in the domain of F.

<u>Proof for $C(X)$</u>. Let $f \in C(X)$. Then $f - f(x_0)1$ vanishes at all the heavy points of F and by Theorem 1, $0 = F(f - f(x_0)1) = F(f) - f(x_0)$. □

<u>Proof for $D(X)$</u>. First let $f \in C(X)$ such that $0 \leq f \leq 1$ and $f(x_0) = 1$. Let $A = f^{-1}(1)$. We claim that $F(k_A) = 1$ where k_A is the characteristic function of A. Assume, to the contrary, that $F(k_A) \neq 1$. Then necessarily $F(k_A) = t < 1$. By the preceding argument, $F(f^n) =$

$f(x_0)^n = 1$ for all indices $n > 0$ and $F(f^n - k_A) = 1 - t > 0$. Put

$$g = \sum_{n=1}^{\infty} (f^n - k_A).$$

By Lemma 5, $g \in D(X)$. But $g \geq \sum_{n=1}^{N} (f^n - k_A)$ for each N and

$$F(g) \geq N(1 - t),$$

which is impossible. So $F(k_A) = 1$.

Now let $h \in D(X)$ with $0 \leq h \leq 1$, and let $h(x_0) > 0$. Pick any $\varepsilon > 0$, so small that $h(x_0) - \varepsilon > 0$. The set $h^{-1}(h(x_0) - \varepsilon, \infty)$ is a class one F_σ-set containing x_0. Thus there is a zero-set A such that $A \subset h^{-1}(h(x_0) - \varepsilon, \infty)$ and $x_0 \in A$. Hence $h \geq (h(x_0) - \varepsilon)k_A$ and $F(k_A) = 1$ by the preceding paragraph. It follows that $F(h) \geq h(x_0) - \varepsilon$. Since ε is arbitrary, we have $F(h) \geq h(x_0)$. On the other hand, if $h(x_0)$ were 0, then we should have $F(h) \geq h(x_0)$ anyway. The same argument with $1-h$ in place of h shows that

$$1-F(h) = F(1-h) \geq (1-h)(x_0) = 1-h(x_0)$$

and $F(h) \leq h(x_0)$. Finally $F(h) = h(x_0)$.

It follows from the preceding paragraph that for any bounded function $h_0 \in D(X)$, $F(h_0) = h_0(x_0)$. Now let q be any function in $D(X)$. By Lemma 1, there is a number $c > 0$ such that $F((q \wedge c) \vee (-c)) = F(q)$ and (by increasing c if necessary) $-c < q(x_0) < c$. It follows that

$$F(q) = F((q \wedge c) \vee (-c)) = ((q \wedge c) \vee (-c))(x_0) = q(x_0). \qquad \square$$

<u>Proof for $E(X)$</u>. Let m be the Baire measure on X found by setting $m(B) = 1$ if $x_0 \in B$ and $m(B) = 0$ if $x_0 \notin B$. Then

$$\int f \, dm = f(x_0)$$

for each $f \in E(X)$ because the set $f^{-1}(f(x_0))$ has measure 1 and its complement has measure 0. Moreover,

$$f(x_0) = F(f) = \int f\, dm$$

for $f \in C(X)$ by the proof for $C(X)$. It follows from Lemmas 3 and 4 that

$$\int f\, dm = F(f)$$

for all $f \in E(X)$. □

Definition. We say that an nlf F is <u>simple</u> if there exist finitely many points $x_1,\dots,x_n \in X$ and positive numbers $a_1,\dots,a_n$ such that $\sum_{j=1}^{n} a_j = 1$ and $F(f) = \sum_{j=1}^{n} a_j f(x_j)$ for all f in the domain of F.

We are now able to characterize all nlfs on $D(X)$ and $E(X)$ when X is realcompact.

Theorem 2. Let F be an nlf on $D(X)$ or $E(X)$, where X is realcompact. Then F is simple.

Proof. By Theorem 1, F has at least one heavy point, and by Lemma 2, F has only finitely many heavy points, $x_1,\dots,x_n \in X$. First we construct a "resolution of the identity" for X. We claim that there exist functions $g_1,\dots,g_n \in C^+(X)$, $0 \leq g_i \leq 1$, for $i = 1,\dots,n$, such that $g_i = 1$ on a nbhd. of x_i and $g_i = 0$ on a nbhd. of x_j for $i \neq j$, and $g_1 + \cdots + g_n = 1$. Let $h_1,\dots,h_n \in C^+(X)$ such that $h_i = 1$ on a nbhd. of x_i for each i, $h_i = 0$ on a nbhd. of x_j for $j \neq i$, and $0 \leq h_i \leq 1$. (Use complete regularity to choose $h \in C(X)$ with $h(x_i) = 2$, $h(x_j) = -1$ for $j \neq i$; put $h_i = (h \vee 0) \wedge 1$.)

If $n = 1$, put $g_1 = 1$. In general $n > 1$ and we suppose that $g_1,\dots,g_{v-1}$ have been constructed such that $g_i = 1$ on a nbhd. of x_i and $g_i = 0$ on a nbhd. of each $x_j (j \neq i)$, $0 \leq g_i \leq 1$, and $g_1 + \cdots + g_{v-1} = 1$. Then the v functions

$$g_1(1-h_v),\ \dots\ ,\ g_{v-1}(1-h_v),\ h_v$$

satisfy the same properties. By induction on v, we obtain the desired functions $g_1, \dots, g_n$.

Each $F(g_i) > 0$ because x_i is a heavy point of F. For each $j = 1, \dots, n$ put $F_j(f) = F(fg_j)/F(g_j)$ for all f in the domain of F. Then each F_j is an nlf with the same domain as F. If $f \in C^+(X)$ and $f(w) > 0$, and w is a heavy point of F_j, then

$$F(f) \geq F(fg_j) = F(g_j) \cdot F_j(f) > 0$$

and w is a heavy point of F. Since the points x_i $(i \neq j)$ cannot be heavy points of F_j (note that $F_j(f) = 0$ for $f \in C^+(X)$ such that f vanishes outside of the nbhd. of x_i where g_j vanishes) it follows that x_j is the only heavy point of F_j. By Lemma 6, $F_j(f) = f(x_j)$ for all $j = 1, \dots, n$, and all f in the domain of F. Moreover, $F(fg_j) = F(g_j)f(x_j)$ and

$$F(f) = F(fg_1+fg_2+\cdots+fg_n) = F(g_1)f(x_1) + F(g_2)f(x_2)+\cdots+F(g_n)f(x_n).$$

Finally, we put $f = 1$ and find that $F(g_1) + F(g_2)+\cdots+F(g_n) = 1$. □

At this juncture we observe that if F is continuous in the topology of pointwise convergence on $C(X)$, then F has at most finitely many heavy points. This is much like Theorem 20 of [3].

Lemma 7. Let F be an nlf on $C(X)$ and let X be realcompact. Then a necessary and sufficient condition that F be continuous on $C(X)$ in the topology of pointwise convergence is that F have only a finite number of heavy points. Moreover, if F is continuous on $C(X)$, then F must be simple on $C(X)$.

Proof. If F has only a finite number of heavy points $x_1,\dots,x_n$, then by the same proof used for Theorem 2, it follows that F is simple. Clearly F is continuous.

Let F be continuous and suppose that there are infinitely many heavy points of F. Let $\{u_1,\dots,u_t\}$ be a (finite) subset of X and let d be a positive number such that if $f \in C(X)$ and $|f(u_j)| < d$ $(j = 1,\dots,t)$, then $|F(f)| < 1$. Take any heavy point w that is different from all the u_j $(j = 1,\dots,t)$. Let $f \in C^+(X)$ such that $f(u_j) = 0$ $(j = 1,\dots,t)$, and $f(w) > 0$. Then $F(f) > 0$ and $F(nf) = nF(f)$ for all integers $n > 0$. Also $nf(u_j) = 0$ for all $j = 1,\dots,t$ and $F(nf) < 1$. But for some n, $nF(f) = F(nf) > 1$ which is impossible. □

We see that $C(X)$ is dense in $D(X)$ and $E(X)$, and $D(X)$ is dense in $E(X)$ in the topology of pointwise convergence. By Theorem 2, any nlf on $D(X)$ or $E(X)$ is continuous. But an nlf F on $C(X)$ need not be continuous. Indeed by Theorem 2 and Lemma 7 it follows that F can be extended to an nlf on $E(X)$ if and only if F is continuous on $C(X)$.

We turn to nonnegative linear functions from $C(X_1)$ to $C(X_2)$.

(We assume all linear functions map the function 1 to the function 1.)

Theorem 3. Let F be a nonnegative linear function mapping $D(X_1)$ into $D(X_2)$ and let X_1 be realcompact. Then F can be extended to a unique nonnegative linear function from $E(X_1)$ to $E(X_2)$.

Proof. For each $y \in X_2$, let F_y be the nlf on $D(X_1)$ defined $F_y(f) = F(f)(y)$ for $f \in D(X_1)$. By Theorem 2, F_y is simple. So there exist finitely many points $x_{yj} \in X_1$ and positive numbers a_{yj} such that $\Sigma_j a_{yj} = 1$ and $F_y(f) = \Sigma_j a_{yj} f(x_{yj})$ for $f \in D(X_1)$.

For each $g \in E(X_1)$, let $F(g)$ be the real function on X_2 defined by $F(g)(y) = \Sigma_j a_{yj} g(x_{yj})$. It remains only to prove that $F(g) \in E(X_2)$; the linearity and nonnegativity of F are evident. Let $\mathcal{B}$ be the family of all functions $g \in E(X_1)$ such that $F(g) \in E(X_2)$. Then $\mathcal{B}$ contains all functions in $C(X_1)$ and indeed in $D(X_1)$. If (g_i) is a sequence of functions in $\mathcal{B}$ converging pointwise to a function $g \in E(X_1)$, then $(F(g_i))$ is a sequence of functions in $E(X_2)$ converging pointwise to $F(g)$. So $F(g) \in E(X_2)$ and $g \in \mathcal{B}$. Finally $\mathcal{B}$ contains $C(X_1)$ and the pointwise limit of any sequence of functions in $\mathcal{B}$ must be in $\mathcal{B}$. So $\mathcal{B} = E(X_1)$. Uniqueness follows from the fact that $D(X_1)$ is dense in $E(X_1)$ and F is plainly continuous on $E(X_1)$. □

Theorem 4. Let F be a nonnegative linear function mapping $C(X_1)$ to $C(X_2)$ and let X_1 be realcompact. Then F can be extended to a nonnegative linear function from $E(X_1)$ to $E(X_2)$ if and only if F is continuous on $C(X_1)$. There is at most one such extension of F.

Proof. Let F be continuous on $C(X_1)$. For each $y \in X_2$, the nlf $f \mapsto F(f)(y)$ is continuous on $C(X_1)$. So $F(f)(y)$ is simple by Lemma 7, and has the form $\sum_j a_j f(x_j)$ (finitely many terms). The proof that F can be extended uniquely to $E(X_1)$ is just like the proof of Theorem 3, so we leave it.

Now let F_0 be an extension of F to $E(X_1)$. Then for each $y \in X_2$, the nlf $f \mapsto F_0(f)(y)$ on $E(X_1)$ must be simple by Theorem 2. It follows that F_0 is continuous on $E(X_1)$, and hence F is continuous on $C(X_1)$. This completes the proof. □

We note also that any nonnegative linear function mapping $D(X_1)$ to $D(X_2)$, or $E(X_1)$ to $E(X_2)$ must be continuous if X_1 is realcompact. Now we draw a conclusion about Baire measures.

Theorem 5. Let m be a Baire measure on a realcompact space X such that $m(X) = 1$ and $C(X) \subset L_1(m)$ (respectively, $D(X) \subset L_1(m)$). Then there is a compact subset Y of X (respectively, finite subset Y of X) such that $m(A) = 0$ for any Baire set $A \subset X\backslash Y$.

Proof. $F(f) = \int_X f\,dm$ is an nlf on $C(X_1)$. By Theorem 1 and its proof, we see that there is a compact set Y composed of all the heavy points of F and a Baire measure m_1 on X such that

$$F(f) = \int_X f\,dm = \int_X f\,dm_1$$

for all $f \in C(X_1)$, and $m_1(A) = 0$ for Baire sets $A \subset X\backslash Y$. By Lemma 3,

$m = m_1$. Finally, if $D(X) \subset L_1(m)$, then F is an nlf on $D(X_1)$, and Y is a finite set by Lemma 2. □

3. <u>Ring homomorphisms</u>. In this Section we consider nlfs on $C(X)$, $D(X)$ and $E(X)$ that are multiplicative; $F(f_1f_2) = F(f_1)F(f_2)$. Such F are obviously ring homomorphisms. Observe that a ring homomorphism F is nonnegative and cannot have more than one heavy point. For if $x_1 \neq x_2$, choose functions $f_1, f_2 \in C^+(X)$ such that $f_1(x_1) > 0$, $f_2(x_2) > 0$, $f_1f_2 = 0$, and note that

$$0 = F(f_1f_2) = F(f_1)F(f_2);$$

then one of the factors on the right must vanish. So if X is a realcompact space, there must be an $x \in X$ such that $F(f) = f(x)$ by Theorem 1 and Lemma 6. (See also [1, 10.5(c)] for $C(X)$.)

Now let X_2 be a completely regular space and X_1 a realcompact space and let F be a ring homomorphism from $C(X_1)$ to $C(X_2)$, or $D(X_1)$ to $D(X_2)$, or $E(X_1)$ to $E(X_2)$. For each $y \in X_2$, there is a point $p(y) \in X_1$ such that the nlf $f \mapsto F(f)(y)$ satisfies $F(f)(y) = f(p(y))$ for all f in the domain of F. Indeed $p(y)$ is unique because $C(X_1)$ separates points in X_1. Moreover,

$$(*) \qquad p^{-1}(f^{-1}(0,\infty)) = (f(p))^{-1}(0,\infty) = (F(f))^{-1}(0,\infty).$$

So if F maps $E(X_1)$ into $E(X_2)$ and A is a Baire set in X_1, let $f \in E(X_1)$ such that $A = f^{-1}(0,\infty)$. Then $p^{-1}(A)$ is a Baire set in X_2 by (*). If F maps $D(X_1)$ into $D(X_2)$ and A is a class one F_σ-set in X_1, let $f \in D(X_1)$ such that $A = f^{-1}(0,\infty)$. Then $p^{-1}(A)$ is a class one F_σ-set in X_2. Finally, if F maps $C(X_1)$ into $C(X_2)$ and A is a cozero-set in X_1, it follows similarly that $p^{-1}(A)$ is a cozero-set in X_2; but the cozero-sets in a completely regular space form a base for the topology, so p is in fact continuous.

Conversely, if p maps X_2 to X_1 such that $p^{-1}(A)$ is a Baire set in X_2 whenever A is a Baire set in X_1, then the mapping $f \mapsto f(p(y))$ $(y \in X_2)$ is a ring homomorphism of $E(X_1)$ to $E(X_2)$. If p maps X_2 to

X_1 such that $p^{-1}(A)$ is a class one F_σ-set in X_2 whenever A is a class one F_σ-set in X_1, then the mapping $f \mapsto f(p(y))$ $(y \in X_2)$ is a ring homomorphism of $D(X_1)$ to $D(X_2)$. (See [2], Theorem 6, p. 143; the arguments there are for the real line, but they also work for general completely regular spaces.) Finally, if p is continuous, then the mapping $f \mapsto f(p(y))$ $(y \in X_2)$ is a ring homomorphism of $C(X_1)$ to $C(X_2)$.

To sum up:

Theorem 6. Let X_2 be a completely regular space and X_1 be a realcompact space. Let F be a ring homomorphism of $C(X_1)$ to $C(X_2)$ (respectively, $D(X_1)$ to $D(X_2)$, $E(X_1)$ to $E(X_2)$). Then there is a function p from X_2 to X_1 such that $F(f)(y) = f(p(y))$ for all $y \in X_2$ and such that $p^{-1}(A)$ is an open set (respectively, class one F_σ-set, Baire set) in X_2 if A is an open set (respectively, class one F_σ-set, Baire set) in X_1. Conversely, if p is such a function from X_2 to X_1, then the mapping F defined by $F(f)(y) = f(p(y))$ for all $y \in X_2$, is such a ring homomorphism. (See also [1, 10.6] for $C(X_i)$.)

In particular, let X_1 and X_2 be realcompact. Then F in Theorem 6 is an isomorphism of $C(X_1)$ onto $C(X_2)$ (respectively, $D(X_1)$ onto $D(X_2)$, $E(X_1)$ onto $E(X_2)$) if and only if p is a one-to-one mapping of X_2 onto X_1 such that p and p^{-1} map open sets to open sets (respectively, class one F_σ-sets to class one F_σ-sets, Baire sets to Baire sets). In this sense, for realcompact X the ring $C(X)$ identifies the open and closed sets in X, the ring $D(X)$ identifies the class one F_σ-sets and G_δ-sets in X, and the ring $E(X)$ identifies the Baire sets in X. The smaller rings contain more information than the larger rings in the sense that the smaller rings identify the more restricted types of sets in X.

On the other hand, an isomorphism between $E(X_1)$ and $E(X_2)$ or $D(X_1)$ and $D(X_2)$, need not map $C(X_1)$ onto $C(X_2)$. Let X_1 be the integers, X_2 the rational numbers, and let p be any bijection of X_2 onto X_1.

We can now use the topology of pointwise convergence on $E(X)$ to determine when ring homomorphisms of $C(X)$ or $D(X)$ or $E(X)$ are isomorphisms.

Theorem 7. Let X_1 and X_2 be realcompact spaces, and let F be a ring homomorphism from $C(X_1)$ to $C(X_2)$ (respectively, $D(X_1)$ to $D(X_2)$, $E(X_1)$ to $E(X_2)$) such that the functions in the image of F separate points in X_2. Then the following are equivalent.

(1) F is a homeomorphism of $C(X_1)$ onto $C(X_2)$ (respectively, $D(X_1)$ onto $D(X_2)$, $E(X_1)$ onto $E(X_2)$),

(2) F maps closed subsets of $C(X_1)$ to closed subsets of $C(X_2)$ (respectively, of $D(X_1)$ to closed subsets of $D(X_2)$, of $E(X_1)$ to closed subsets of $E(X_2)$),

(3) F is a ring isomorphism of $C(X_1)$ onto $C(X_2)$ (respectively, $D(X_1)$ onto $D(X_2)$, $E(X_1)$ onto $E(X_2)$).

Proof. We will give the proof only for $C(X)$. The proofs for $D(X)$ and $E(X)$ are analogous.

(1) => (2). Clear.

(2) => (3). We must prove that F is a bijective mapping. Suppose, to the contrary, there is a nonzero $f \in C^+(X)$ such that $F(f) = 0$ in $C(X_2)$. For each integer n, let $f_n = nf + (n^{-1})1$. Then $F(f_n) = n^{-1}1 \in C(X_2)$ and the set $F\{f_n\}$ is not a closed subset of $C(X_2)$. But for some $x \in X_1$, $f(x) > 0$ and $f_n(x) \to \infty$, and this implies that the set $\{f_n\}$ has no accumulation point in $C(X_1)$. Thus $\{f_n\}$ is a closed subset of $C(X_1)$ that does not map to a closed subset of $C(X_2)$. This contradiction proves that F is one-to-one.

Now suppose $y_1,\ldots,y_j \in X_2$. Let p be the function in Theorem 6. The points $p(y_1),\ldots,p(y_j)$ are distinct because $F(C(X_1))$ separates points in X_2. If $a_1,\ldots,a_j$ are any real numbers, we can use complete regularity to find an $f \in C(X_1)$ such that $f(p(y_i)) = a_i$ for $i = 1,\ldots,j$. Then $F(f)(y_i) = a_i$ for $i = 1,\ldots,j$. Thus $F(C(X_1))$ is dense in $C(X_2)$. But $F(C(X_1))$ is also closed in $C(X_2)$ by (2), so $F(C(X_1)) = C(X_2)$. This proves (3).

(3) => (1). By Theorem 6 (and the discussion following it), there exists a one-to-one function p mapping X_2 onto X_1 such that $F(f)(y) = f(p(y))$ for $y \in X_2$ and $f \in C(X_1)$. Thus if U is an open subset of the real line, for $y \in X_2$,

$$F\{f \in C(X_1) : f(p(y)) \in U\} = \{g \in C(X_2) : g(y) \in U\}.$$

So F and F^{-1} map subbasic open sets to subbasic open sets. Since F is a bijective mapping onto $C(X_2)$, if follows that F and F^{-1} map open sets to open sets. This proves (1). □

In Theorem 6 we cannot expect $p(X_2) = X_1$ even when F is one-to-one. For example, let X_1 be the compact interval $[0,1]$, let X_2 be the open interval $(0,1)$, and let p be the inclusion mapping of X_2 into X_1. Then F is one-to-one, but $p(X_2) \neq X_1$. We offer

Theorem 8. Let F be a one-to-one linear mapping from $C(X_1)$ to $C(X_2)$ that maps $C^+(X_1)$ into $C^+(X_2)$ and let X_1 and X_2 be realcompact. Then a sufficient condition that p of Theorem 6 satisfy $p(X_2) = X_1$ is that for each $s \in S \setminus X_2$, where S is the Stone-Čech compactification of X_2, there is a $g \in C(X_1)$ with

$$\lim_{x \to s} (F(g))(x) = \infty.$$

Proof. Assume this condition. Any nonempty cozero-set U in X_1 meets $p(X_2)$; for if $\{x \in X_1 : f(x) \neq 0\} \cap p(X_2) = \emptyset$, then $F(f) = 0 = F(0)$, contrary to the hypothesis that F is one-to-one.

Fix $w \in X_1$. We will prove that $w \in p(X_2)$. Suppose, to the contrary, $w \notin p(X_2)$. Let $\mathcal{E}$ denote the family of all sets of the form $U \cap p(X_2)$ where U is a cozero-set in X_1. It follows that every set in $\mathcal{E}$ is nonvoid and the intersection of any two sets in $\mathcal{E}$ is also in $\mathcal{E}$. The family of sets $\{p^{-1}(A) : A \in \mathcal{E}\}$ has the same property. So there is a point $s \in S$ such that s is in the closure of $p^{-1}(A)$ for each $A \in \mathcal{E}$.

We claim that $s \notin X_2$. For suppose $s \in X_2$. Then $p(s) \neq w$. Choose an $f \in C(X_1)$ such that $f(w) = 1$ and $f(p(s)) = 0$. Put $A = f^{-1}(\frac{1}{2},2) \cap p(X_2)$. Then $A \in \mathcal{E}$ and this set is disjoint from the open set $f^{-1}(-1,\frac{1}{2})$. Also $p(s) \in f^{-1}(-1,\frac{1}{2})$ and $s \in p^{-1}(f^{-1}(-1,\frac{1}{2}))$. But p is continuous, so $p^{-1}(f^{-1}(-1,\frac{1}{2}))$ is a nbhd. of s that is disjoint from $p^{-1}(A)$. And s is in the closure of $p^{-1}(A)$. This contradiction proves that $s \in S \setminus X_2$.

By hypothesis there is a $g \in C(X_1)$ such that

$$\lim_{x \to s} (F(g))(x) = \infty.$$

Let r be any number greater than $g(w)$. But $B = g^{-1}(-\infty,r) \cap p(X_2)$. Then $B \in \mathcal{E}$ and $p^{-1}(B)$ is disjoint from the set $(F(g))^{-1}(r,\infty) = p^{-1}(g^{-1}(r,\infty))$. In S, $(F(g))^{-1}(r,\infty)$ is a nbhd. of s that is disjoint from $p^{-1}(B)$, and this is impossible. □

The condition given in Theorem 8 is sufficient to make $p(X_2) = X_1$, but it is not necessary. Let X_1 be the compact interval $[0,1]$ and let X_2 be the discrete space $[0,1]$. Let p be the identity mapping on X_2. Then $p(X_2) = X_1$, but any function in $F(C(X_1))$ is bounded.

4. Metrizable spaces. Let F be an nlf on $C(X)$ and let X be realcompact. Then there is a compact subset Y of X such that $f \in C(X)$ and $f(Y) = 0$ imply that $F(f) = 0$ (Theorem 1). Now let $\{U\}$ be an open covering of X. There exist finitely many sets $U_1,\dots,U_n$ in the covering such that $Y \subset U_1 \cup \cdots \cup U_n$. So $f \in C(X)$ and $f(U_1 \cup \cdots \cup U_n) = 0$ imply that $F(f) = 0$. This inspires the following definition.

Definition. Let F be an nlf on $C(X)$ and let X be completely regular and Hausdorff. We say that F is regular if for each open covering $\{U\}$ of X, there exist finitely many sets $U_1,\dots,U_n$ in $\{U\}$ such that $f \in C(X)$ and $f(U_1 \cup \cdots \cup U_n) = 0$ imply $F(f) = 0$.

Thus every nlf on $C(X)$ is regular if X is realcompact. We will show that the converse statement is true for metrizable spaces: if every nlf on $C(X)$ is regular and if X is metrizable, then X is realcompact.

Theorem 9. Let F be a regular nlf on $C(X)$ and let X be metrizable. Then F has at least one heavy point.

Proof. Let ρ be an appropriate metric on X. Assume, to the contrary, that F has no heavy point. The family of open balls $S(x,1)$ $(X \in X)$ covers X. Let $h \in C^+(X)$ such that $0 \leq h \leq 1$ and $F(h) > 0$. For example, 1 is such a function. Since f is regular, there are finitely many points $x_1,\ldots,x_n \in X$ such that any $g \in C(X)$ coinciding with h on $S(x_1,1) \cup \cdots \cup S(x_n,1)$ must satisfy

$$|F(g) - F(h)| = |F(g-h)| = 0 \quad \text{and} \quad F(g) = F(h) > 0.$$

For each $j = 1,\ldots,n$, let $g_j \in C^+(X)$ such that $0 \leq g_j \leq 1$, $g_j = 1$ on $S(x_j,1)$ and $g_j = 0$ outside of $S(x_j,2)$. Then $h(g_1 \vee \cdots \vee g_n)$ coincides with h on $S(x_1,1) \cup \cdots \cup S(x_n,1)$, and it follows that

$$F(h\cdot(g_1 \vee \cdots \vee g_n)) = F(h) > 0$$

and

$$F(hg_1) + \cdots + F(hg_n) \geq F(h\cdot(g_1 \vee \cdots \vee g_n)) > 0.$$

For some j, $F(hg_j) > 0$. For this j, set $h_1 = g_j$ and $u_1 = x_j$. Then $F(hh_1) > 0$, $0 \leq h_1 \leq 1$, $h_1 = 1$ on $S(u_1,1)$ and h_1 vanishes outside of $S(u_1,2)$.

By argument in the preceding paragraph, with hh_1 in place of h, there is a function $h_2 \in C^+(X)$ such that $F(hh_1h_2) > 0$, $0 \leq h_2 \leq 1$, and a point $u_2 \in X$ such that $h_2 = 1$ on $S(u_2,\frac{1}{2})$ and h_2 vanishes outside of $S(u_2,1)$. Likewise there is an $h_3 \in C^+(X)$ such that $F(hh_1h_2h_3) > 0$, $0 \leq h_3 \leq 1$, and a point $u_3 \in X$ such that $h_3 = 1$ on $S(u_3,\frac{1}{4})$ and h_3 vanishes outside of $S(u_3,\frac{1}{2})$.

In general, there is an $h_j \in C^+(X)$ such that $F(hh_1\cdots h_j) > 0$, $0 \leq h_j \leq 1$, and a point $u_j \in X$ such that $h_j = 1$ on $S(u_j,2^{1-j})$ and h_j vanishes outside of $S(u_j,2^{2-j})$. Since $hh_1\cdots h_j$ is positive at some point t,

$$\rho(u_{j-1},u_j) \leq \rho(u_{j-1},t) + \rho(u_j,t) < 2^{3-j} + 2^{2-j} < 2^{4-j}.$$

It follows that the sequence of points (u_j) is a Cauchy sequence in X.

We claim that the Cauchy sequence (u_j) does not converge in X. Suppose, to the contrary, (u_j) converges to $u \in X$. Then u is a light point of F, and there is a $p \in C^+(X)$ such that $p(u) > 0$ and $F(p) = 0$. Let $c > 0$ be a number so large that $cp(u) > 1$. Say $cp > 1$ on the open nbhd. V of u. For large enough j, $S(u_j,2^{2-j}) \subset V$ and it follows that $0 \leq h_j < cp$. But $F(cp) = cF(p) = 0$, so $F(h_j) = 0$. Finally, $0 \leq hh_1\cdots h_j \leq h_j$ and hence $F(hh_1\cdots h_j) = 0$, which is impossible. Hence

(u_j) does not converge in X.

Any $x \in X$ has a nbhd. U such that all but finitely many h_j vanish on U. Put $k_j = hh_1 \cdots h_j$ for each $j \geq 1$. Then $F(k_j) > 0$ and $\sum_{j=1}^{\infty} k_j/F(k_j)$ sums to a function $k \in C^+(X)$. For each integer $N > 0$,

$$k \geq \sum_{j=1}^{N} k_j/F(k_j)$$

and

$$F(k) \geq F(\sum_{j=1}^{N} k_j/F(k_j)) = N,$$

which is impossible. □

As we noted in Section 3, a ring homomorphism F from $C(X)$ to the real numbers can have at most one heavy point. We can draw some conclusions about ring homomorphisms on $C(X)$, $D(X)$ and $E(X)$ when X is metrizable.

<u>**Theorem 10**</u>. Let X be a metrizable space and let F be a ring homomorphism of $C(X)$ or $D(X)$ or $E(X)$ such that the restriction of F to $C(X)$ is regular. Then there is a point $x_0 \in X$ such that $F(f) = f(x_0)$ for all f in the domain of F.

<u>Proof for $C(X)$</u>. Let x_0 be the unique heavy point of F. Choose any function $f \in C(X)$ such that x_0 is not in the closure of the set $\{x : f(x) \neq 0\}$. Let $g \in C^+(X)$ such that $g(x_0) > 0$ and $fg = 0$. Then

$$0 = F(fg) = F(f)F(g),$$

and since x_0 is a heavy point of F, $F(g) > 0$ and $F(f) = 0$. But if $f_0 \in C^+(X)$ and $f_0(x_0) = 0$, then for any number $\varepsilon > 0$ we have

$$F((f_0 \vee \varepsilon) - \varepsilon 1) = 0 = F(f_0 \vee \varepsilon) - \varepsilon.$$

(Here put $f = (f_0 \vee \varepsilon) - \varepsilon 1$ in the preceding argument.) Hence

$$0 \leq F(f_0) = F((f_0 \vee \varepsilon) - \varepsilon 1) + F((f_0 \wedge \varepsilon)) = F((f_0 \wedge \varepsilon)) \leq \varepsilon.$$

Since ε is arbitrary, $F(f_0) = 0$.

Thus if $h \in C(X)$ and $h(x_0) = 0$, we obtain $F(h \vee 0) = F(h \wedge 0) = 0$, and $F(h) = 0$ also. For any $q \in C(X)$, $q - (q(x_0))1$ vanishes at x_0 and

$$F(q) - q(x_0) = F(q - (q(x_0)1) = 0.$$

<u>Proof for D(X)</u>. Let g denote the characteristic function of the singleton set $\{x_0\}$. Then $g \in D(X)$, and $0 \leq F(g) \leq F(1) = 1$. We claim that $F(g) = 1$. Suppose, to the contrary, that $F(g) < 1$. Let $f \in C^+(X)$ such that $0 \leq f \leq 1$, and $f = 1$ at x_0 and at no other point. Then $F(f) = 1$ and $F(f-g) = 1 - F(g) > 0$. By Lemma 5,

$$\sum_{n=1}^{\infty} (f^n - g) \in D(X).$$

For any index $N \geq 1$, $\sum_{n=1}^{\infty} (f^n - g) \geq \sum_{n=1}^{N} (f^n - g)$ and

$F(\sum_{n=1}^{\infty} (f^n - g)) \geq F(\sum_{n=1}^{N} (f^n - g)) = \sum_{n=1}^{N} (1 - F(g)) = N(1 - F(g))$, which is impossible. This proves $F(g) = 1$.

Take any $h \in D(X)$ satisfying $h(x_0) = 0$. Then

$$F(h) = F(h\cdot(1 - g)) = F(h)(1 - F(g)) = 0.$$

So for $q \in D(X)$,

$$0 = F(q - (q(x_0))1) = F(q) - q(x_0).$$

The proof for $E(X)$ is analogous to the proof for $D(X)$, so we leave it. □

It follows from Theorems 9 and 10 that if every ring homomorphism F from $C(X)$ to the reals is regular, and if X is metrizable, then X is realcompact. We now have a number of conditions equivalent to realcompactness for metrizable spaces.

Theorem 11. Let X be a metrizable space. Then the following are equivalent.

(1) X is realcompact.

(2) Every nlf on C(X) is regular.

(3) Every ring homomorphism from C(X) to the reals is regular.

(4) Every ring homomorphism from C(X) to the reals has a heavy point.

Proof. (1) => (2) follows from Theorem 1 and remarks at the beginning of this Section. (2) => (3) is clear. (3) => (4) follows from Theorem 9. (4) => (1) is just like the proof of Theorem 10 in the case C(X), so we leave it. □

We do not know if there exists a metrizable space that is not realcompact, but we close with this observation.

Theorem 12. Let (X,ρ) be a metric space that is not realcompact. Then there is a ring homomorphism F from C(X) to the reals such that for some $c > 0$, there do not exist finitely many open balls $B_1,\ldots,B_n$ in X, each of radius c, for which $f \in C(X)$ and $f(B_1 \cup \cdots \cup B_n) = 0$ imply $F(f) = 0$.

Proof. By Theorem 11, there is a ring homomorphism F from C(X) to the reals that has no heavy point. The condition must hold for F, for otherwise the argument in the proof of Theorem 9 would go through and F would have a heavy point. We leave the rest. □

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