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APPROXIMATE PEANO DERIVATIVES AND THE BAIRE* ONE PROPERTY

A real valued function f defined on the real line $\mathbb{R}$ is said to have an approximate Peano derivative of order k at x if there are finite numbers $f_{(0)}(x)$, $f_{(1)}(x)$, ..., $f_{(k)}(x)$, and a set E of density one at zero such that

$$\text{(A)} \qquad f(x_o + h) - \sum_{i=0}^{k} \frac{f_{(i)}(x)}{i!} h^i = o(h^k) \quad \text{as } h \to 0,\ h \in E.$$

In this paper we shall insist that $f_{(0)}(x) = f(x)$ so that the notion of approximate continuity at x will correspond to the notion of having an approximate Peano derivative of order 0 at x. If one replaces the expression $o(h^k)$ in (A) by $O(h^k)$, the resulting weaker property is called approximate Peano boundedness of order k at x, thereby paralleling the terminology used by Ash [2] for Peano differentiability and Peano boundedness of order k.

Approximate Peano derivatives are known to share many of the properties of ordinary derivatives and papers investigating these properties include references [4] through [11]. The purpose of this note is to present a proof, using only first principles, that if a function is approximately Peano bounded of order $k + 1$ at each real number, then the k^{th} approximate Peano derivative of the function belongs to the class Baire* one in the notation of

[12] (or class [C] in the notation of [1].) The proof takes advantage of a function sequence construction originally utilized by the present author [6] to show that approximate Peano derivatives are in class Baire one and the following two elementary lemmas, the first due to Auerbach [3] and the second being a well known exercise in mathematical induction.

LEMMA A. *If* $\Sigma\varphi_n$ *is a series of continuous functions on* $\mathbb{R}$ *and* Σa_n *is a convergent series of positive constants such that for each* $x \in \mathbb{R}$ *there is a positive number* $N(x)$ *with the property that* $|\varphi_n(x)| \le a_n$ *whenever* $n \ge N(x)$, *then for each nonempty closed set* F *there is an open interval* I *such that* $I \cap F$ *is not empty and* $\Sigma\varphi_n$ *converges uniformly on* $I \cap F$ *(and, consequently,* $\Sigma\varphi_n$ *is in class Baire** *one.)*

LEMMA B. *For any real number* λ

$$\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}\left(\lambda + j - \frac{k}{2}\right)^i = 0, \quad i = 0, 1, \ldots, k-1$$

$$= k!, \quad i = k.$$

The symbol $\Delta_k(x,h;f)$ will be used to denote the Riemann difference

$$\Delta_k(x,h;f) = \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} f(x + jh - \tfrac{1}{2}kh)$$

THEOREM. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be approximately Peano bounded of order* $k+1$ *at each* $x \in \mathbb{R}$. *Then the function* $f_{(k)} : \mathbb{R} \to \mathbb{R}$ *belongs to class Baire** *one.*

Proof. Note first of all that according to the aforementioned Lemma A of Auerbach, it will suffice to find the existence of a sequence $\{\phi_n\}$ of continuous functions on $\mathbb{R}$ such that for each x there is a number $B(x)$ and a natural number $N(x)$ such that

$$(1) \qquad |\phi_n(x) - f_{(k)}(x)| \le B(x)/2^n \quad \text{for} \quad n > N(x).$$

Specifically then, we could apply Auerbach's lemma to the series $\phi_1 + \sum_{n=1}^{\infty} (\phi_{n+1} - \phi_n)$ to conclude that $f_{(k)} = \lim_{x \to \infty} \phi_n$ is Baire* one. Consequently, the remainder of this proof will consist of the construction of the sequence $\{\phi_n\}$ and the verification of (1).

For each positive integer n, each integer p, each nonzero real number h, and each real number α, set

$$I_{n,p} = [(p - \tfrac{3}{2})/2^n,\ (p + \tfrac{3}{2})/2^n], \qquad I_n = [-1/2^{n+1},\ 1/2^{n+1}],$$

$$S_{n,p,\alpha,h} = \{x \in I_{n,p} : \Delta_k(x,h;f)/h^k > \alpha\},$$

$$T_{n,p,\alpha} = \{\tfrac{1}{2}kh \in I_n : |S_{n,p,\alpha,h}| > \tfrac{1}{2}|I_{n.p}|\},$$

and at each point of the form $p/2^n$, define

$$\phi_n(p/2^n) = \sup\{\alpha : |T_{n,p,\alpha}| > \tfrac{1}{2}|I_n|\}.$$

Finally, extend ϕ_n linearly to arrive at a continuous function on all of $\mathbb{R}$.

Let $x_o \in \mathbb{R}$. There is a number $C(x_o)$ such that the set

$$E = \{h : \left| f(x_o + h) - \sum_{i=0}^{k} \frac{f_{(i)}(x_o)}{i!} h^i \right| < C(x_o)|h|^{k+1}\}$$

has density one at zero. Next, set $B(x_o) = 7^{k+1}(2k)^k C(x_o)$. We shall show that (1) will hold with this choice for $B(x_o)$.

Let ϵ be a positive number less than $1/4(k+1)$. There is a positive number δ such that $|E \cap I| > (1 - \epsilon)|I|$ for any interval I containing 0 of length less than δ. Choose a positive integer $N(x_o)$ so large that $1/2^{N(x_o)} < \delta/4$.

Let $n > N(x_o)$ and select the unique integer p so that $p/2^n < x_o \le (p+1)/2^n$. Next, let h be any number such that

$$\tfrac{1}{2}kh \in [-1/2^{n+1}, -1/2^{n+3}] \cup [1/2^{n+3}, 1/2^{n+1}]$$

and hold it fixed. For each $j = 0, 1, \ldots k$ let $B_j = \{y - jh + \tfrac{1}{2}kh : y \in E\}$. Then for each $j = 0, 1, \ldots, k$ we have

$$\frac{|B_j \cap [-3/2^{n+1}, 1/2^{n+1}]|}{1/2^{n-1}} > 1 - \epsilon,$$

and so, letting $B = \bigcap_{j=0}^{k} B_j$, it follows that

$$\frac{|B \cap [-3/2^{n+1}, 1/2^{n+1}]|}{1/2^{n-1}} > 1 - (k+1)\epsilon > 3/4.$$

Furthermore, if $\lambda h \in B \cap [-3/2^{n+1}, 1/2^{n+1}]$, then $x_o + \lambda h \in I_{n,p}$ and $\lambda h + jh - \tfrac{1}{2}kh \in E$ for each $j = 0, 1, \ldots, k$. The next immediate goal will be to show that

(2) $$|\Delta_k(x_o + \lambda h, h; f)/h^k - f_{(k)}(x_o)| < B(x_o)/2^n.$$

We have

(3) $$|\Delta_k(x_o + \lambda h, h; f)/h^k - f_{(k)}(x_o)| =$$

$$= \left|\frac{1}{h^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x_o + \lambda h + jh - \tfrac{1}{2}kh) - f_{(k)}(x_o)\right|$$

$$\le \left|\frac{1}{h^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left[f(x_o + \lambda h + jh - \tfrac{1}{2}kh) - \sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}(\lambda + j - \tfrac{1}{2}k)^i h^i\right]\right| +$$

$$+ \left|\frac{1}{h^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}(\lambda + j - \tfrac{1}{2}k)^i h^i - f_{(k)}(x_o)\right|.$$

However,

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}(\lambda+j-\frac{k}{2})^i h^i = \sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}h^i\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\lambda+j-\frac{k}{2})^i$$

$$= f_{(k)}(x_o)h^k,$$

where the last equality is due to Lemma B..

Consequently, the second absolute value on the rightmost side of inequality (3) is identically zero, yielding

$$(4)\quad |\Delta_k(x_o+\lambda h,h;f)/h^k - f_{(k)}(x_o)| \le$$

$$\le \left|\frac{1}{h^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left[f(x_o+\lambda h+jh-\tfrac{1}{2}kh) - \sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}(\lambda+j-\tfrac{1}{2}k)^i h^i\right]\right|.$$

However, for each $j = 0, 1, \dots k$, $\lambda h + jh - \frac{1}{2}kh \in E$ and hence

$$\left|f(x_o+\lambda h+jh-\tfrac{1}{2}kh) - \sum_{i=0}^{k}\frac{f_{(i)}(x_o)}{i!}(\lambda+j-\tfrac{1}{2}k)^i h^i\right| <$$

$$< C(x_o)|\lambda+j-\tfrac{1}{2}k|^{k+1}|h|^{k+1}$$

$$\le C(x_o)|h|^{k+1}(|\lambda|+\tfrac{1}{2}k)^{k+1}$$

$$\le C(x_o)|h|^{k+1}\left[\frac{3}{2^{n+1}|h|}+\frac{1}{2}k\right]^{k+1}$$

$$\le C(x_o)|h|^{k+1}\left[\frac{3k\cdot 2^{n+2}}{2^{n+1}}+\frac{1}{2}k\right]^{k+1}$$

$$< C(x_o)(7k|h|)^{k+1}.$$

Incorporating this estimate in inequality (4), we obtain

$$(5)\quad |\Delta_k(x_o+\lambda h,h;f)/h^k - f_{(k)}(x_o)| < C(x_o)(7k)^{k+1}|h|\sum_{j=0}^{k}\binom{k}{j}$$

$$= C(x_o)(7k)^{k+1}2^k|h|$$

$$\le C(x_o)(7k)^{k+1}2^k\cdot\frac{1}{2^n k}$$

$$= \frac{B(x_o)}{2^n},$$

thereby establishing inequality (2).

Let $W_{x_o,h,n} = \{x \in I_{n,p} : |\Delta_k(x,h;f)/h^k - f_{(k)}(x_o)| < B(x_o)/2^n\}$. To this point we have shown that for a fixed number $\frac{1}{2}kh \in [-1/2^{n+1}, -1/2^{n+3}] \cup [1/2^{n+3}, 1/2^{n+1}]$, we have $|W_{x_o,h,n}| > \frac{3}{4}\cdot\frac{1}{2^{n-1}}$. Consequently,

$$|\{\tfrac{1}{2}kh \in I_n : |W_{x_o,h,n}| > \tfrac{3}{4}\cdot\tfrac{1}{2^{n-1}}\}| > \tfrac{3}{4}|I_n|,$$

and so

$$|\{\tfrac{1}{2}kh \in I_n : |W_{x_o,h,n}| > \tfrac{1}{2}|I_{n,p}|\}| > \tfrac{3}{4}|I_n|.$$

This, together with the definition of $\phi_n(p/2^n)$, implies that

$$f_{(k)}(x_o) - B(x_o)/2^n \le \phi_n(p/2^n) \le f_{(k)}(x_o) + B(x_o)/2^n,$$

and this inequality is valid for all $n > N(x_o)$. In a similar manner we can show that for $n > N(x_o)$ we have

$$f_{(k)}(x_o) - B(x_o)/2^n \le \phi_n((p + 1)/2^n) \le f_{(k)}(x_o) + B(x_o)/2^n.$$

Therefore, for $n > N(x_o)$ $|\phi_n(x_o) - f_{(k)}(x_o)| \le B(x_o)/2^n$, establishing the validity of inequality (1) and completing the proof.

An immediate consequence of this theorem is, of course, the fact that if a function has a (finite) Peano derivative of order $k + 1$ at each point of the real line, then its Peano derivative of order k is a Baire* one function, a result first proved by Denjoy [5].

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