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A DERIVATIVE OFTEN ZERO AND DISCONTINUOUS

Let D_0 denote the set of all bounded derivatives on $[0,1]$ that vanish on a dense subset of $[0,1]$. Then D_0 is a complete metric space [3] under the sup metric. A slight modification of an argument by Clifford Weil [4], shows that the set of all derivatives in D_0 that are discontinuous almost everywhere on $[0,1]$ is a residual subset of D_0. In [1], it is shown that the apparently smaller set of all derivatives in D_0 that are nonzero almost everywhere on $[0,1]$ is a residual subset of D_0. The question arises whether these sets really do differ. Are there derivatives in D_0 that are discontinuous almost everywhere on $[0,1]$ and yet vanish on a set of positive measure? In any case, the set of all such derivatives is only a first category subset of D_0. In this note we construct such a derivative directly.

We construct a derivative $h \in D_0$ that is discontinuous almost everywhere on $[0,1]$ and yet vanishes on a set of positive measure in each subinterval of $[0,1]$.

Note that any such derivative necessarily is nonzero on a first category set of positive measure in each subinterval of $[0,1]$. We begin our construction with a derivative in D_0 that is nonzero almost everywhere. Let f_0 be a bounded nonnegative derivative on $[0,1]$ that vanishes at each rational point and is positive on a dense set of irrational points [2], [5]. Let $f_1(x) = f_0(x)$ for $0 \leq x \leq 1$, and in general make f_1 periodic on R with period 1. Put

$$f(x) = \Sigma_{n=1}^{\infty} 2^{-n} f_1(2^n x) \qquad (0 \leq x \leq 1),$$

Then $f \in D_0$ and $f(0) = f(1) = 0$. Let m denote Lebesgue measure. Then $m\{x \in (0,1): f_1(x) > 0\} > 0$; otherwise the indefinite integral of f_1 would be constant on $(0,1)$. Let $m\{x \in (0,1): f_1(x) > 0\} = \epsilon > 0$. Routine arguments show that $m\{x \in I: f(x) > 0\} \geq \epsilon m(I)$ for any interval I. Thus the set $f^{-1}(0,\infty)$ has no point of dispersion in $[0,1]$ and hence $m(f^{-1}(0,\infty)) = 1$.

In the rest of this note, we assume that $f \in D_0$, $0 \leq f \leq 1$, $f(0) = f(1) = 0$, and $f > 0$ almost everywhere on $[0,1]$. Let X denote the (dense) set of points where f is continuous. Then f vanishes on X.

Lemma 1. Let $U \subset (0,1)$ be an open set, dense in $(0,1)$, such that $R\backslash U$ is a perfect set, $f\chi_U \in D_0$. Then there exists an open set $V \subset U$, dense in $(0,1)$, such that $R\backslash V$ is a perfect set, $f\chi_V \in D_0$, and such that for each component interval I of U, $m(I \cap V) < \frac{1}{2}m(I)$.

Proof. Let I be a component interval of U. By induction, we construct a sequence of mutually disjoint, nonabutting, open subintervals $J_1, J_2, J_3, \ldots$ of I with endpoints in X, such that $\sup f(J_n) < n^{-1}$, $m(J_n) < 2^{-n-1}m(I)$ for each n, and $\cup_{n=1}^{\infty} J_n$ is dense in I. (To do this, construct J_n around a point $x_n \in X$ where f is continuous and 0.) Now Let $I_1, I_2, I_3, \ldots$ be the component intervals of U, and for each i let $J_{i1}, J_{i2}, J_{i3}, \ldots$ be the open intervals chosen in this way for the component I_i. Put $V = \cup_{ij} J_{ij}$. Then $R\backslash V$ is a perfect set, V is evidently open and dense in each I_i and hence dense in $(0,1)$. Also for each i,

$$m(V \cap I_i) = \Sigma_j\, m(J_{ij}) < \Sigma_j\, 2^{-j-1}m(I_i) = \tfrac{1}{2}m(I_i).$$

It remains only to prove that $f\chi_V$ is the derivative of its indefinite integral F. We prove that $F'_+(x) = f(x)\chi_V(x)$ for $0 \leq x < 1$. The proof that $F_-'(x) = f(x)\chi_V(x)$ for $0 < x \leq 1$ is analogous. There are three cases to consider.

1. Suppose $x \in V$ or x is the left endpoint of some J_{ij}. Then the conclusion is clear because f is a derivative.

2. Suppose $x \in [0,1]\setminus U$. Let G be the indefinite integral of $f\chi_U \in D_0$. Then for $t > x$, we have

$$0 \leq F(t) - F(x) = \int_x^t f\chi_V \leq \int_x^t f\chi_U = G(t) - G(x).$$

But $$G_+'(x) = f(x)\chi_U(x) = 0$$ and it follows that
$$F_+'(x) = f(x)\chi_V(x) = 0.$$

3. Suppose $x \in U\setminus V$ and x is not the left endpoint of any J_{ij}. Say $x \in I_i$. For $t \in I_i$ and $t > x$ we have

$$0 \leq F(t) - F(x) = \int_x^t f\chi_V = \Sigma_* \int_{J_{ij}\cap(x,t)} f \leq \Sigma_* j^{-1}m(J_{ij}\cap(x,t))$$

where Σ_* means sum on those j for which J_{ij} meets the interval (x,t). But the intervals J_{ij} are mutually disjoint, and it follows that $0 \leq F(t) - F(x) \leq k^{-1}(t-x)$ where k is the smallest index j for which J_{ij} meets (x,t). Consequently

$$\lim_{t\to x+} (F(t)-F(x))(t-x)^{-1} = 0 = F_+'(x) = f(x)\chi_V(x). \qquad \square$$

Put $U_0 = (0,1)$. Note that $f = f\chi_{U_0}$. In general, let the open set U_{n+1} be obtained from U_n the same way V was obtained from U in Lemma 1. Then $m(U_{n+1}) < \frac{1}{2}m(U_n)$ for each n. Moreover, $U_0 \supset U_1 \supset U_2 \supset U_3 \supset \ldots$ is a contracting sequence of open subsets of $(0,1)$ such that $m(\cap_{n=0}^{\infty} U_n) = 0$, $f\chi_{U_n} \in D_0$ and $m(I\setminus U_{n+1}) > 0$ for each component interval I of U_n for each n. For $n \geq 0$, define $f_n = f\chi_{U_n} - f\chi_{U_{n+1}}$.

Let I be a component interval of U_n. Then $m(I\setminus U_{n+1}) > 0$. So there is an $x \in I\setminus U_{n+1}$ such that $f(x) = f_n(x) > 0$. Select an interval J contained in I with $x \in J$ and endpoints in X such that $m(J) < d^2$, where d is the distance between the interval J and the set $R\setminus I$.

Let $I_1, I_2, I_3, \ldots$ be the component intervals of U_n. For each i, let J_i be a subinterval of I_i the same way J is a subinterval of I in the preceding paragraph. We define the function g_n as follows:

$$\text{for } x \in J_i, \quad g_n(x) = f_n(x)/\sup f_n(J_i),$$

$$\text{for } x \in [0,1]\setminus \cup_i J_i, \quad g_n(x) = 0.$$

Lemma 2. For $n \geq 1$, $g_n \in D_0$, g_n vanishes on $U_{n-1}\setminus U_n$ and g_n is discontinuous almost everywhere on $U_{n-1}\setminus U_n$.

Proof. Obviously $0 \leq g_n \leq 1$. By construction, f_n vanishes on $R\setminus U_n$, so g_n vanishes on $U_{n-1}\setminus U_n$. Now take a point $x \in U_{n-1}\setminus U_n$; necessarily x is not an isolated point of $U_{n-1}\setminus U_n$. Since U_n is dense in U_{n-1}, there will be component intervals of U_n in every neighborhood of x. But in each component interval I of U_n, $\sup g_n(I) = 1$ by construction. Thus g_n is discontinuous at x. It follows that g_n is discontinuous almost everywhere on $U_{n-1}\setminus U_n$.

It remains only to prove that g_n is the derivative of its indefinite integral F. We will only prove that $F'_+(x) = g_n(x)$ for $0 \leq x < 1$. The proof of $F'_-(x) = g_n(x)$ for $0 < x \leq 1$ is analogous.

If $x \in$ any I_i or is the left endpoint of any I_i (where I_i is a component interval of U_n), the conclusion is clear because f_n is a derivative. Suppose that $x \in [0,1]$ is not such a point, and let J_i be the subinterval of I_i used in the definition of g_n. Take $t > x$. Then

$$0 \le F(t) - F(x) = \int_x^t g_n \le \Sigma_* \, m(J_i) \le \Sigma_* \, m(I_i \cap (x,t))^2$$

where Σ_* means sum over those i for which J_i meets the interval (x,t). But the intervals I_i are mutually disjoint, so

$$0 \le F(t) - F(x)_* \le \Sigma \;\; m(I_i \cap (x,t))^2 \le (t-x)^2$$

and clearly

$$\lim_{t \to x+}(F(t)-F(x))(t-x)^{-1} = 0 = F'_+(x) = g_n(x). \qquad \square$$

Put

$$h = \Sigma_{j=0}^{\infty} 2^{-j}(f_{2j+1} + g_{2j+1}).$$

Then $h \in D_0$ because the functions f_{2j+1} and g_{2j+1} are functions in D_0 bounded by 0 and 1. Let K be any open subinterval of $(0,1)$. Then K contains some component interval I of U_{2n} for some n. All the functions f_{2j+1} and g_{2j+1}, and h as well, vanish on $U_{2n} \backslash U_{2n+1}$ and hence on $I \backslash U_{2n+1}$. But $m(I \backslash U_{2n+1}) > 0$ by construction. It remains only to prove that h is discontinuous almost everywhere on $(0,1)$.

As we just saw, h vanishes on $U_{2n} \backslash U_{2n+1}$ for any $n \ge 0$. Also $h \ge 2^{-n} g_{2n+1}$ and it follows that h must be discontinuous at any point in $U_{2n} \backslash U_{2n+1}$ where g_{2n+1} is discontinuous. By Lemma 2, g_{2n+1} and h are discontinuous almost everyhwere on $U_{2n} \backslash U_{2n+1}$. On $U_{2n+1} \backslash U_{2n+2}$ for $n \ge 0$, we have $h \ge 2^{-n} f_{2n+1} = 2^{-n} f$, and $f > 0$ almost everywhere on $U_{2n+1} \backslash U_{2n+2}$. Thus $h > 0$ almost everywhere on $U_{2n+1} \backslash U_{2n+2}$. But h is discontinuous at any point where h is positive because h vanishes on a dense subset of $[0,1]$. It follows that h is discontinuous almost everywhere on $U_{2n+1} \backslash U_{2n+2}$. Recall that $m(\cap_{n=0}^{\infty} U_n) = 0$. Finally, h is discontinuous almost everywhere on

$$(0,1) = \left(\cup_{n=o}^{\infty} (U_n \backslash U_{n+1})\right) \cup \left(\cap_{n=o}^{\infty} U_n\right).$$

We began with a function f obtained from [2], and constructed h from f. A topic for further research could be to seek a metric on D_0 (or on some other set of functions) that would allow us to prove the existence of such derivatives by a category argument.

References

1. F.S. Cater, Two large subsets of a function space, International Journal of Mathematics and Mathematical Sciences, 8(1985), pp. 189-191.

2. Y. Katznelson & K. Stromberg, Everywhere differentiable, nowhere monotone functions, Amer. Math. Monthly, 81(1974), pp. 349-354.

3. Clifford E. Weil, On nowhere monotone functions, Proceedings Amer. Math. Society, 56(1976), pp. 388-389.

4. Clifford E. Weil, The space of bounded derivatives, Real Analysis Exchange, 3(1977-78), pp. 38-41.

5. Z. Zahorski, Sur la premiere derivee, Trans. Amer. Math. Society, 69(1950), pp. 1-54.