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CONNECTIVITY FUNCTIONS WITH A PERFECT ROAD

Let X and Y be topological spaces. A function $f:X \to Y$ is said to be a connectivity function provided that if A is a connected subset of X, then the graph of f restricted to A is a connected subset of $X \times Y$. A function $f:X \to Y$ is said to be an almost continuous function provided that if O is an open subset of $X \times Y$ containing the graph of f, then there exists a continuous function $g:X \to Y$ such that O contains the graph of g. A real-valued function f defined on an interval is said to have a perfect road at the point x provided that there exists a perfect set P such that x is a bilateral point of accumulation of P and such that f restricted to P is continuous at x.

Let I be the closed unit interval. In order that a function $f:I \to I$ be a connectivity function, it is necessary and sufficient that the graph of the entire function be connected. However, if $f:I^2 \to I$ and the entire graph is connected it is not guaranteed that f is a connectivity function. Also, if $f:I \to I$ is an almost continuous function, then f is a connectivity function. But there exist connectivity functions $f:I \to I$ that are not almost continuous, [3], [7], [10]. However, if $f:I^2 \to I$ is a connectivity function, then f is an almost continuous function, [11], but the converse is not true. Using these facts a negative answer was given to the question posed by Stallings, "Can a connectivity function $f:I \to I$ be extended to a connectivity function $I^2 \to I$ when I is embedded in I^2 as $I \times \{0\}$?" Thus if $f:I \to I$ is a connectivity function, having f an almost continuous function is a necessary condition to insure that f can be extended to a connectivity function $I^2 \to I$. However, it has been shown by Gibson and Roush that having f an almost continuous function is not a sufficient condition, [5].

The purpose of this paper is to prove the following theorem.

<u>Theorem</u>. Let $f:I \to I$ be a function and let $g:I^2 \to I$ be an extension of f. If g is a connectivity function, then f has a perfect road at each point.

<u>Proof</u>. Choose any $x \in I$ that is not an endpoint. We now show that there exists a perfect subset A of I to the left of x and containing x such that f restricted to A is continuous at x.

If for some $\varepsilon > 0$, f is constant on the interval $(x-\varepsilon, x]$, then we may let $A = [a,x]$ where $a \in (x-\varepsilon, x)$. Otherwise, assume that f is not constant on any interval $(x-\varepsilon, x]$.

Let $\varepsilon_1 > 0$. By theorem 1 of [6], there exists $x_1 \in (x-\varepsilon_1, x)$ such that $|f(x_1) - f(x)| < \varepsilon_1$ and $f(x_1) \neq f(x)$. By theorem 2 of [5], there exists a Cantor set $C_1 \subset (x_1, x)$ such that $f(C_1)$ is between $f(x_1)$ and $f(x)$; and moreover, f restricted to C_1 is continuous. Let $\varepsilon_2 \leq d(C_1, x)$ where d is the usual distance function. As above there exists $x_2 \in (x-\varepsilon_2, x)$ such that $|f(x_2) - f(x)| < \varepsilon_2$ and $f(x_2) \neq f(x)$. Also there exists a Cantor set $C_2 \subset (x_2, x)$ such that $f(C_2)$

is between $f(x_2)$ and $f(x)$; and moreover, f restricted to C_2 is continuous.

By induction construct a sequence ε_n such that ε_n converges to 0 and a sequence of Cantor sets C_n such that $\varepsilon_n \leq d(C_{n-1}, x)$ and f restricted to C_n is continuous. Note that C_i and C_j are disjoint whenever $i \neq j$. Then $A = (\bigcup_{n=1}^{\infty} C_n) \cup \{x\}$ is a Cantor set to the left of x containing x such that f restricted to A is continuous at x.

Likewise we can show that there is a perfect subset B of I to the right of x containing x such that f restricted to B is continuous at x.

Then $P = A \cup B$ is a perfect road for f at x. In each case, when we deal with the end points of I, the bilateral condition is replaced with a unilateral condition.

Notice that we have actually proved that f restricted to P is continuous at each point of P.

The first example given by Gibson and Roush, [4], is an almost continuous function $f: I \to I$ that has a perfect road at no point of I and can not be extended to a connectivity function. This leads to a natural question.

Question 2. Does there exist an almost continuous function $f: I \to I$ that has a perfect road at each point but can not be extended to a connectivity function $I^2 \to I$?

It is a well-known fact that if $f:I \to I$ is a Baire class 1 function, then the following statements are equivalent, [2].

(1) f has a connected graph, [8].

(2) f is an almost continuous function, [1].

(3) f has a perfect road at each point, [9].

This leads to another natural question.

<u>Question 2</u>. Does there exist a Baire class 1 connectivity function $f:I \to I$ that can not be extended to a connectivity function $I^2 \to I$?

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