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Some Properties of Semi-continuous Functions

Introduction. The purpose of this paper is to answer some questions posed by Ceder and Pearson [6] in a recent survey paper. In particular, it is proved that to each bounded upper semi-continuous function f, defined on a closed interval I, there corresponds a bounded Darboux upper semi-continuous function g, also defined on I, which differs from f on a first category subset of I having Lebesgue measure zero; moreover, we will show that g may be chosen so that $f \leq g$ on I.

We will also show that, if f and g are two Darboux upper semi-continuous functions defined on a closed interval I and $f < g$ on I, there exists a Darboux upper semi-continuous function h defined on I such that $f < h < g$ on I.

We conclude this paper by showing that some of the properties which are known to be "typical" in the family of Darboux functions in Baire class one are also "typical" in the family of Darboux upper semi-continuous functions.

Since a function f is upper semi-continuous if and only if -f is lower semi-continuous, results concerning upper semi-continuous functions yield corresponding results for lower semi-continuous functions. Analogues of the above mentioned results for the family of bounded Darboux Baire 1 functions may be found in [4], [5], [8] and [9].

Notation and Terminology. Given a family of functions which forms a Banach Space, a property will be called "typical" in the family if it holds for all functions except those in some first category subset of the space.

We denote by usc and $\mathcal{B}_1$ the set of functions, defined on $[0,1]$, which are upper semi-continuous and in Baire class 1, respectively. The families b$\mathcal{D}$usc and b$\mathcal{D}\mathcal{B}_1$ denote the corresponding bounded functions having the Darboux (i.e.intermediate value) property. These families form Banach spaces with norm $||f||=\sup|f|$. Moreover, each of the families b$\mathcal{D}$usc and b$\mathcal{D}\mathcal{B}_1$ is a closed subspace of $\mathcal{B}_1$. The sets $C(f)$ and $A(f)$ denote, respectively, the points of continuity and approximate continuity of the function f.

The interval $[0,1]$ will be denoted by I, and for any function f defined on I we take $Q^+(f)$ (resp. $Q^-(f)$) to be the set of points x in I such that $f(x) = \limsup_{z\to x+} f(z)$ (resp. $f(x) = \limsup_{z\to x-} f(z)$).

A. A Relationship Between Semi-continuous and Darboux Semi-continuous Functions

We begin with two lemmas which will be used in the proofs of some of the results appearing in this and later sections.

Lemma A1. *Let* $f \in$ busc. Then $I \setminus [Q^+(f) \cap Q^-(f)]$ *is a denumerable set.*

Lemma A2. A function $f \in \mathcal{D}usc$ if and only if $I \subset Q^{+}(f) \cap Q^{-}(f)$.

Lemma A1 is a consequence of [7] Lemma 4. The proof of Lemma A2 is a direct consequence of the definition of a Darboux upper semi-continuous function defined on I. Basic to the proof of the main result of this section is

Lemma A3. Let f be a bounded upper semi-continuous function and let $\{P_n\}_{n=1}^{\infty}$ be a sequence of non-empty pairwise disjoint, perfect subsets of I such that each P_n is a nondenumerable Borel set and

(a) if $a_n = \sup P_n$ then $f(x) \leq f(a_n)$ for all $x \in P_n$,

(b) $\operatorname{diam} P_n < 1/n$.

Then there exists a bounded upper semi-continuous function g such that

(1) $\{x: f(x) \neq g(x)\} \subset \bigcup_{n=1}^{\infty} P_n$,

(2) $f(x) \leq g(x)$ for all x in I,

(3) if $f(P_n) \subset [y_n, f(a_n)]$ for some y_n then $g(P_n) \subset [y_n, f(a_n)]$,

(4) $\bigcup_{n=1}^{\infty}\{a_n\} \cup Q^{-}(f) \subset Q^{-}(g)$ and $Q^{+}(f) \subset Q^{+}(g)$.

Proof. Since f is bounded we may assume, without loss of generality, that $0 \leq f(x) < M$ for some real number M and all x in I.

Fix n. Let $\{F_m\}_{m=1}^{\infty}$ be a sequence of non-empty disjoint perfect subsets of P_n defined in the following manner: for $m = 1$, let r_1 be such that $0 \leq a_n - r_1 < 1$, and let F_1 be a non-empty perfect subset of $(a_n - r_1, a_n) \cap P_n$. Assuming that $F_1, F_2, \ldots, F_m$ have been chosen so that F_i is a non-empty perfect subset of P_n and $F_i \cap F_j = \emptyset$ whenever $i \neq j$ and $i,j \in \{1,2,\ldots,m\}$ then define F_{m+1} by setting $r_{m+1} = \min\{1/m+1, \operatorname{dist}(F_m, a_n)\}$ and taking F_{m+1} to be a non-empty perfect subset of $(a_n - r_{m+1}, a_n) \cap P_n$. Clearly, the sequence $\{F_m\}_{m=1}^{\infty}$ defined in this way consists of non-empty pairwise disjoint perfect subsets of P_n and the set $\{a_n\} \cup \bigcup_{m=1}^{\infty} F_m$ is closed.

For each $m \geq 1$, let H_m be a bilaterally c-dense-in-itself F_σ subset of F_m and let $h_m \in \mathcal{D}usc$ be such that $h_m(I) = [0, f(a_n)]$ and $I \setminus H_m = h_m^{-1}\{\{0\}\}[1]$.

Setting $h = \sum_{m=1}^{\infty} h_m$, then it is clear that h is Darboux and upper semi-continuous on $I \setminus \{a_n\}$ and $h(\bigcup_{m=1}^{\infty} H_m) = (0, f(a_n)]$.

Let $g_n = \max\{f, h\}$. Clearly, g_n is a bounded function satisfying properties (1) through (3). Moreover, since $f \in$ usc, it follows, by (1) through (3) above, that $\{x\colon g_n(x) \geq t\}$ is closed for all real numbers $t \geq f(a_n)$. Furthermore, by the definition of the sequences $\{h_m\}_{m=1}^{\infty}$ and $\{F_m\}_{m=1}^{\infty}$, $\{x\colon g_n(x) \geq t\}$ is also closed when $0 \leq t \leq f(a_n)$. Therefore, $g_n \in$ usc.

We now show that $\{a_n\} \cup Q^-(f) \subset Q^-(g_n)$. Let x be any point of this set and consider the following three cases.

<u>case</u> 1. $a_n = x$. Then $g_n(x) = f(x)$. By the definition of $\{h_m\}_{m=1}^{\infty}$, we can choose a sequence $\{x_m\}_{m=1}^{\infty}$ increasing to a_n with $x_m \in H_m$ for each m, and $h_m(x_m) = f(a_n)$; moreover,

$$\underline{\lim_{m\to\infty}}\, g_n(x_m) \geq \underline{\lim_{m\to\infty}}\, h_m(x_m) = \underline{\lim_{m\to\infty}}\, f(x) = f(x) = g_n(x).$$

Therefore, $a_n \in Q^-(g_n)$.

<u>case</u> 2. $a_n \neq x$ and $g_n(x) = f(x)$. Since $x \in Q^-(f)$, there exists a sequence $\{x_m\}_{m=1}^{\infty}$ increasing to x such that $\limsup_{m\to\infty} f(x_m) = f(x)$. Thus, if $x \notin Q^-(g_n)$ then

$$g_n(x) > \limsup_{m\to\infty} g_n(x_m) \geq \limsup_{m\to\infty} f(x_m) = f(x) = g_n(x),$$

a contradiction. Hence, $x \in Q^-(g_n)$ when $f(x) = g_n(x)$.

<u>case</u> 3. $a_n \neq x$ and $g_n(x) = h(x)$. Since h is a Darboux upper semi-continuous function on $I\backslash\{a_n\}$ it follows that $x \in Q^-(h)$. Hence, there exists a sequence $\{x_m\}_{m=1}^{\infty}$ increasing to x such that $\limsup_{m\to\infty} h(x_m) = h(x)$. Then, if $x \notin Q^-(g_n)$ we get

$$g_n(x) > \limsup_{m\to\infty} g_n(x_m) \geq \limsup_{m\to\infty} h(x_m) = h(x) = g_n(x),$$

a contradiction, giving $x \in Q^-(g_n)$.

Consequently, $Q^-(f) \subset Q^-(g_n)$. Similarly, $Q^+(f) \subset Q^+(g_n)$.

For each n, let g_n be the function obtained as above and define

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in P_n, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, g is a bounded function satisfying properties (1) through (3) above. Let us show that $g \in$ usc.

If $g \notin$ usc, then there exists an x and a sequence $\{x_k\}_{k=1}^{\infty}$ converging to x such that $\limsup_{k\to\infty} g(x_k) > g(x)$. If $g(x_k) = f(x_k)$ for all but finitely many k then

$$g(x) \geq f(x) \geq \limsup_{k\to\infty} f(x_k) = \limsup_{k\to\infty} g(x_k) > g(x),$$

a contradiction, giving $g \in$ usc. Thus, we may assume, without loss of generality, that $g(x_k) \neq f(x_k)$ and $x_k \in P_k$ for all k.

By (1) and (3) above, $f(a_k) = g_k(a_k) \geq g_k(x_k) = g(x_k)$ for all k. Since $x_k \to x$ and diam $P_k \to 0$ as $k \to \infty$, it follows that $a_k \to x$. Hence,

$$g(x) \geq f(x) \geq \limsup_{k\to\infty} f(a_k) \geq \limsup_{k\to\infty} g(x_k) > g(x),$$

a contradiction. Therefore, g is upper semi-continuous.

Finally, it remains to show that g satisfies property (4). Clearly, $a_n \in Q^-(g)$ for every n. Hence, it is enough to show that $Q^-(f) \subset Q^-(g)$ and $Q^+(f) \subset Q^+(g)$. Since the proofs are similar we will only prove that $Q^-(f) \subset Q^-(g)$.

To do this, let $x \in Q^-(f)$ be such that $x \neq a_n$ for all n. Either $g(x) = f(x)$ or $g(x) = g_n(x)$ for some n.

If $g(x) = f(x)$ then, since $x \in Q^-(f)$, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ increasing to x such that $\limsup_{k\to\infty} f(x_k) = f(x)$. Thus, if $x \notin Q^-(g)$ we get

$$g(x) > \limsup_{k\to\infty} g(x_k) \geq \limsup_{k\to\infty} f(x_k) = f(x) = g(x),$$

a contradiction leading to $Q^-(f) \subset Q^-(g)$. A similar argument holds if $g(x) = g_n(x)$. Hence, in either case $Q^-(f) \subset Q^-(g)$. This completes the proof of this lemma.

<u>Theorem</u> A1. <u>If</u> f <u>is a bounded upper semi-continuous function, then there exists a bounded Darboux upper semi-continuous function</u> g <u>such that</u> $\{x: f(x) \neq g(x)\}$ <u>is a first category null subset of</u> I, <u>and</u> $f(x) \leq g(x)$ <u>for all</u> x <u>in</u> I.

<u>Proof</u>. Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of $I \backslash Q^-(f)$ (Lemma A1). Since $f \in$ usc, there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of open subintervals of I such that $a_n = \sup I_n$, $\operatorname{diam} I_n < 1/n$ and

$f(x) < f(a_n)$ for all $x \in I_n$. Clearly, we can choose, for each n, sets $P_n \subset I_n$ such that the sequence $\{P_n\}_{n=1}^{\infty}$ consists of non-empty, disjoint, nowhere dense and null perfect subsets of I satisfying (a) and (b) of Lemma A3. Hence, there exists a function $h \in$ busc satisfying (1) through (4) of Lemma A3. By property (2) we have $f(x) \leq h(x)$ for all $x \in I$ and property (4) implies that $I \subset Q^-(h)$.

Using a similar argument applied to the function h and $I \setminus Q^+(h)$, we get the function g of this theorem.

B. Insertion of Darboux Semi-continuous Functions

Theorem 1 of [5] states that if two Darboux Baire 1 functions f and g, defined on I, are such that $f < g$ on I then there exists a Darboux Baire 1 function h defined on I such that $f < h < g$ on I. In Theorem B2 below we shall show that the analogous result holds for two Darboux upper semi-continuous functions.

It is a simple matter to show that the average of two Darboux upper semi-continuous functions need not even be Darboux. This is given in the following.

<u>Example</u> B1. Let $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(x) = 1$ if $x = 0$. Let $g(x) = -\sin(1/x)$ if $x \neq 0$ and $g(x) = 1$ if $x = 0$. Both f and g are Darboux and upper semi-continuous, but $h = ½(f+g)$ is zero when $x \neq 0$ and 1 when $x = 0$. Thus, h is not Darboux.

To prove the main theorem of this section we need

<u>Lemma</u> B1. <u>Let</u> f <u>be a bounded upper semi-continuous function and let</u> g <u>be a bounded Darboux Baire 1 function</u>. <u>Suppose that</u> $f(x) < g(x)$ <u>for all</u> x <u>in</u> I. <u>Then there exists a bounded Darboux upper semi-continuous function</u> h <u>such that</u> $f(x) \le h(x) < g(x)$ <u>for all</u> x <u>in</u> I.

<u>Proof</u>. First, we show that there exists $h_1 \in usc$ such that

(1) $f(x) \le h_1(x) < g(x)$ for all x in I,

(2) $Q^-(h_1) = I$ and $Q^+(f) \subset Q^+(h_1)$.

To do this, let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of $I \setminus Q^-(f)$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of open subintervals of I such that $a_n = \sup I_n$, $\operatorname{diam} I_n < 1/n$ and $f(x) < f(a_n)$ for all x in I_n. We shall construct a sequence $\{P_n\}_{n=1}^{\infty}$ of pairwise disjoint perfect sets satisfying the following two properties

(c) $a_n = \sup P_n$ and $f(x) < f(a_n) < g(x)$ for all $x \in P_n$,

(d) $\operatorname{diam} P_n < 1/n$.

The sequence $\{P_n\}_{n=1}^{\infty}$ may be chosen as follows. First, let $n = 1$. Since $f(a_1) < g(a_1)$, there is an $r_1 > 0$ such that $f(a_1) + r_1 < g(a_1)$. Since $g \in \mathcal{DB}_1$, there exists a nonempty perfect set $P_1 \subset I_1$ such that $a_1 = \sup P_1$, $f(a_1) + r_1 < g(x)$ if $x \in P_1$ and $a_j \notin P_1$ if $j > 1$. Assuming that pairwise disjoint sets $P_1, P_2, \ldots, P_n$ have been chosen so that P_i is perfect and $a_j \notin P_i$ if $j > i$, $1 \le i \le n$. To define P_{n+1}, note that $f(a_{n+1}) < g(a_{n+1})$, so there exists an $r_{n+1} > 0$ such that $f(a_{n+1}) + r_{n+1} < g(a_{n+1})$. Since $g \in \mathcal{DB}_1$ there is a perfect set

$P_{n+1} \subset I_{n+1} \setminus \bigcup_{i=1}^{n} P_i$ such that $a_{n+1} = \sup P_{n+1}$ and $f(a_{n+1}) + r_{n+1} < g(x)$ for all $x \in P_{n+1}$. Clearly, the sequence $\{P_n\}_{n=1}^{\infty}$ chosen in this way satisfies properties (a) and (b) of Lemma A3. Hence, there exists a bounded upper semi-continuous function h_1 satisfying (1) through (4) of Lemma A3. The relation $h_1 < g$ on I was built into the construction. This together with (2) of Lemma A3 implies that h_1 satisfies (1) above. Moreover, (4) of Lemma A3 implies that h_1 also satisfies (2) above.

Applying a similar argument to h_1 and $I \setminus Q^+(h_1)$ we get the function h of this lemma, completing the proof.

Theorem B1. Let f be a bounded upper semi-continuous function and let g_1 and g_2 be two bounded Darboux Baire 1 functions such that $g_1 < f < g_2$ on I. Then there exists a bounded Darboux upper semi-continuous function g such that $g_1 < g < g_2$ on I.

Proof. By Lemma B1, there exists $g \in b\mathcal{D}usc$ such that $f \leq g < g_2$ on I. Clearly, g is the desired function.

Theorem B2. Let g_1 and g_2 be two bounded Darboux upper semi-continuous functions such that $g_1 < g_2$ on I. Then there exists a bounded Darboux upper semi-continuous function g such that $g_1 < g < g_2$ on I.

Proof. Since $g_1, g_2 \in usc$, the function $f = \frac{1}{2}(g_1+g_2) \in usc$ and $g_1 < f < g_2$ on I. By Theorem B1, there exists $g \in b\mathcal{D}usc$ such that $f \leq g < g_2$ on I. Clearly, g is the desired function.

Remark B1. Let f be a real valued function defined on I and let P be a nonempty perfect subset of I. We write $f \in \mathcal{D}(P)$ if whenever x is a right (resp. left) limit point of P, f(x) is a right (resp. left) limit point of f(P). We write $f \in \mathcal{B}_1(P)$ if $f^{-1}(G) \cap P$ is an F_σ for each open set G. These definitions are due to Bruckner and Ceder [3]. Note that if P is an interval, $\mathcal{D}(P) \cap \mathcal{B}_1(P)$ coincides with the Darboux functions on P [2].

We will say that f is upper semi-continuous on P, written $f \in usc(P)$, if $f|P$, the restriction of f to P, is upper semi-continuous on P.

It is clear that, with proper modifications, all the lemmas and theorems we have established are extendable to functions $f \in usc(P)$. In particular we have

Theorem B3. *Let P be a nonempty perfect subset of I, or an interval, and let $f \in busc(P)$ (resp. $f \in blsc(P)$). Then there exists a function $g \in b\mathcal{D}usc(P)$ (resp. $g \in b\mathcal{D}lsc(P)$) such that $\{x: f(x) \neq g(x)\}$ is of first category in P and of Lebesgue measure zero.*

Theorem B4. *Let P be a nonempty perfect subset of I, or an interval, and let $h_1, h_2 \in b\mathcal{D}usc(P)$ (resp. $h_1, h_2 \in b\mathcal{D}lsc(P)$) be such that $h_1 < h_2$ on P. Then there exists a function $g \in b\mathcal{D}usc(P)$ (resp. $g \in b\mathcal{D}lsc(P)$) such that $h_1 < g < h_2$ on P.*

C. Additional Properties of Darboux Semi-continuous Functions

In Theorems C2 through C6 below we consider the nature of the dense sets in $b\mathcal{D}$usc. We will show that some of the properties, known to be typical in $b\mathcal{DB}_1$, are also typical in the family $b\mathcal{D}$usc. First we prove

<u>Lemma</u> C1. <u>Let</u> f <u>be a bounded upper semi-continuous function and let</u> $\varepsilon > 0$. <u>Then there exists a bounded upper semi-continuous function</u> g <u>having finite range such that</u> $||f - g|| < \varepsilon$.

<u>Proof</u>. Since f is bounded there exists a real number M such that $-M < f(x) < M$ for all x in I.

Let $a_1, a_2, \ldots, a_n$ be a sequence of points in $[-M, M]$ such that $-M = a_n < a_{n-1} < \ldots < a_2 < a_1 = M$ and $a_i - a_{i+1} < \varepsilon$ for all $i \in \{1, 2, \ldots, n-1\}$.

For each $i \geq 1$ set $G_i = \{x: f(x) \geq a_i\}$ and if $1 \leq i \leq n-1$ set $A_i = G_{i+1} \setminus G_i$. Now, define $g(x) = a_i$ if $x \in A_i$.

Clearly, $I = \bigcup_{i=1}^{n-1} A_i$ and the range of g is contained in $\{a_1, a_2, \ldots, a_{n-1}\}$. Hence, g is a bounded function having finite range. Moreover, for any real number t, if $T = \{i: t \geq a_i\}$, then $\{x: g(x) \geq t\} = \bigcup_{i \in T} \{x: f(x) \geq a_i\}$ which is obviously closed. Therefore, g is also upper semi-continuous, and by the choice of $a_1, a_2, \ldots, a_n$, $||f - g|| < \varepsilon$. This completes the proof.

Lemma C2. Let f be a bounded upper semi-continuous function and let $\varepsilon > 0$. Suppose that there exists a bounded Darboux Baire 1 function h such that $||f - h|| < \varepsilon$. Then there exists a bounded Darboux upper semi-continuous function g such that

(1) $\{x: f(x) \neq g(x)\}$ is a null subset of I,

(2) $||f - g|| < 16\varepsilon$,

Proof. To begin with, assume that f has finite range. Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of $I \backslash Q^-(f)$. Since $h \in \mathcal{DB}_1$, for each $n \geq 1$ there exists a perfect set P_n such that $a_n = \sup P_n$ and $h(P_n) \subset (h(a_n)-\varepsilon, h(a_n)+\varepsilon)$. Moreover, since $f \in$ usc and $||f - h|| < \varepsilon$, it follows that $f(P_n) \subset (f(a_n)-3\varepsilon, f(a_n)]$.

Clearly, the sets $\{P_n\}_{n=1}^{\infty}$ may be chosen so that each P_n is null, $\operatorname{diam} P_n < 1/n$, and $P_n \cap P_m = \emptyset$ if $n \neq m$. That is, the sequence $\{P_n\}_{n=1}^{\infty}$ satisfies properties (a) and (b) of Lemma A3. Hence, there exists a function $k \in$ busc satisfying (1) through (4) of Lemma A3. Since $f(P_n) \subset (f(a_n)-3\varepsilon, f(a_n)]$ we have $||f - k|| < 3\varepsilon$ and $||h - k|| \leq 4\varepsilon$. Now, if g is the function obtained by applying the above argument to k and $I \backslash Q^+(k)$, then g satisfies (1) above and $||k - g|| < 12\varepsilon$. It follows that $||f - g|| < ||f - k|| + ||k - g|| \leq 15\varepsilon$. Finally, if f does not have finite range, we can choose f_1 of finite range such that $||f - f_1|| < \varepsilon$, in which case the resulting function g satisfies (2) above.

Theorem C1. The class of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ (resp. $f(C(f))$) is finite is dense in $b\mathcal{D}usc$.

Proof. Let $f \in b\mathcal{D}usc$ and $\varepsilon > 0$. Using Lemma C1, we can obtain a function $h \in busc$ having finite range such that $||f - h|| < \varepsilon/16$. Now, apply Lemma C2 to get a function $g \in b\mathcal{D}usc$ such that $||f - g|| < \varepsilon$ and $g = h$ a.e. It is easy to verify that $g(A(g))$ is finite. Since $C(g) \subset A(g)$, $g(C(g))$ is also finite.

The proofs of Theorems C2 through C6 below are identical to those in the case of $b\mathcal{D}\mathcal{B}_1$. Some indication of the proof is given in each case. For detailed proofs see [8].

Theorem C2. The class of functions $f \in b\mathcal{D}usc$ such that $cl f(A(f))$ (resp., $cl f(C(f))$) has Lebesgue measure zero is a residual G_δ in $b\mathcal{D}usc$.

Proof. By Theorem C1, the class, $\mathcal{H}$, of functions $f \in b\mathcal{D}usc$ such that $cl f(A(f))$ has Lebesgue measure zero is dense in $b\mathcal{D}usc$. Thus, it remains to show that $\mathcal{H}$ is a G_δ set. This is a consequence of the fact that $\mathcal{H} = b\mathcal{D}usc \setminus \bigcup_{n=1}^{\infty} F_n$, where F_n is the closed subset of $b\mathcal{D}usc$ consisting of $\{f: |cl f(A(f))| \geq 1/n\}$. Hence, Theorem C2 is true "$A(f)$". A similar argument holds for "$C(f)$".

Theorem C3. The class of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ (resp., $f(C(f))$) is nowhere dense and null is a residual in $b\mathcal{D}usc$.

Theorem C3 is a direct consequence of Theorem C2 since the class of Theorem C3 contains the residual class of Theorem C2.

Theorem C4. The class of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ (resp., $f(C(f))$) is nowhere dense is a residual G_δ in $b\mathcal{D}usc$.

Proof. By Theorem C1, the class, $\mathcal{H}$, of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ is nowhere dense is dense in $b\mathcal{D}usc$. Hence, we only need to show that $\mathcal{H}$ is a G_δ set. Let $\{J_n\}_{n=1}^\infty$ be an enumeration of all rational intervals.

Let $E_n = \{f \in b\mathcal{D}usc\colon J_n \subset cl f(A(f))\}$. The set E_n is closed in $b\mathcal{D}usc$ and $\mathcal{H} = b\mathcal{D}usc \setminus \bigcup_{n=1}^\infty E_n$. Hence, the theorem follows for "$A(f)$". The proof for "$C(f)$" is similar.

Theorem C5. The class of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ (resp. $f(C(f))$) is countable is a dense first category subset of $b\mathcal{D}usc$.

Proof. By Theorem C1, the class, $\mathcal{H}$, of functions $f \in b\mathcal{D}usc$ such that $f(C(f))$ is countable is dense in $b\mathcal{D}usc$. Let $\{J_n\}_{n=1}^\infty$ be an enumeration of all rational intervals.

Let $A_n = \{f \in b\mathcal{D}usc\colon f \text{ is constant on } J_n \cap C(f)\}$. Each A_n is closed and nowhere dense; moreover, $\mathcal{H} = \bigcup_{n=1}^\infty A_n$. Hence, $\mathcal{H}$ is of first category. If $\mathcal{F}$ denotes the class of functions $f \in b\mathcal{D}usc$ such that $f(A(f))$ is countable then $\mathcal{F} \subset \mathcal{H}$. Hence, $\mathcal{F}$ is also of first category and dense (by Theorem C1).

Many other properties are typical in b$\mathcal{D}$usc and b$\mathcal{D}$lsc see [10], however these properties can be proved in a more general setting [11].

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