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A RIESZ-TYPE DEFINITION OF THE DENJOY INTEGRAL

Riesz [4] defines a Lebesgue integrable function as the almost everywhere limit of a mean convergent sequence of step functions. A short proof of the uniqueness of the definition can be found in [2]. In this note we give a similar definition for the Denjoy integral and show that using this definition a convergence theorem can be proved.

First, we give some definitions [6]. Let X be a closed set in [a,b]. A function F is said to be absolutely continuous in the restricted sense on X or $AC_*(X)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever

$$\sum_i |b_i - a_i| < \delta$$

where $[a_i, b_i]$, $i = 1,2,\ldots$, is a finite or infinite sequence of nonoverlapping intervals in [a,b] and $a_i, b_i \in X$ for all i, we have

$$\sum_i \omega(F; [a_i, b_i]) < \varepsilon$$

where ω denotes the oscillation of F over $[a_i, b_i]$. Then F is ACG_* if [a,b] is the union of closed sets X_i, $i = 1,2,\ldots$, such that F is $AC_*(X_i)$ for each i. A function f is *Denjoy integrable* on [a,b] if there exists a continuous and ACG_* function F such that the derivative $F'(x) = f(x)$ almost everywhere in [a,b].

Next, a sequence of functions f_n is said to be *control-convergent* to f on [a,b] if the following conditions are satisfied :

(i) $f_n(x) \to f(x)$ almost everywhere in [a,b] as $n \to \infty$ and each f_n is Denjoy integrable on [a,b];

(ii) the primitives F_n of f_n are ACG_* uniformly in n, i.e., [a,b] is the union of closed sets X_i on each of which F_n is $AC_*(X_i)$ uniformly in n;

(iii) $F_n(x)$ converges uniformly on [a,b] as $n \to \infty$.

We define a RD integrable function f on [a,b] to be the limit almost everywhere of a control-convergent sequence of step functions ϕ_n, and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b \phi_n(x)\,dx .$$

We shall see that the integral is uniquely determined.

<u>CONTROLLED CONVERGENCE THEOREM</u> *If* f_n *is control-convergent to* f *on* [a,b], *then* f *is Denjoy integrable on* [a,b] *and*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

The proof is given in [3]. Bullen pointed out to the authors that the convergence theorem is also proved in [1; p.50 Theorem 47] with (iii) replaced by

(iv) F_n are equicontinuous in [a,b].

In fact, the two sets of conditions are equivalent. Suppose that conditions (i), (ii) and (iv) hold. We claim that $\{F_n(x)\}$ is bounded at every point $x \in [a,b]$. Indeed, in view of (iv), for every $x \in [a,b]$ there exists $\delta(x) > 0$ such that

$$|F_n(x) - F_n(y)| \leq 1 \quad \text{for every } n$$

whenever $|x-y| < \delta(x)$. Then it follows from the Heine-Borel covering theorem that there exists a finite number of points, say, $x_1, x_2, \ldots, x_N$, such that the union of $(x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1,2,\ldots,N$ covers $[a,b]$. For any $y \in [a,b]$ we have $y \in (x_i - \delta(x_i), x_i + \delta(x_i))$ for some i and

$$\begin{aligned} |F_n(y)| &\leq |F_n(y) - F_n(x_i)| + |F_n(x_i)| \\ &\leq 1 + (2i-1) \\ &\leq 2N \end{aligned}$$

Hence $\{F_n(x)\}$ is uniformly bounded and therefore bounded at each x.

By Ascoli's theorem [5; p.155], the above sequence $\{F_n\}$ has a subsequence which converges pointwise uniformly on $[a,b]$. In view of the controlled convergence theorem, the function f is Denjoy integrable and this subsequence converges to F, the primitive of f. Consequently, for every subsequence of $\{F_n\}$, there exists a subsubsequence which converges uniformly to F on $[a,b]$. Therefore condition (iii) holds by reductio ad absurdum. The converse is easy.

In other words, the controlled convergence theorem also follows from [1] without reference to [3]. As a corollary of the controlled convergence theorem, we have the following.

UNIQUENESS THEOREM *If a sequence of step functions* ϕ_n *is control-convergent to zero on* $[a,b]$, *then*

$$\lim_{n\to\infty} \int_a^b \phi_n(x)dx = 0.$$

The theorem can also be proved directly. In view of its similarity to [3], we shall not reproduce the proof.

Next, we show that the Denjoy and RD integrals are equivalent. It is easy to see from the controlled convergence theorem that every RD integrable function is Denjoy integrable.

Now suppose f is Denjoy integrable on [a,b]. We shall prove that it is RD integrable there. Let F be the primitive of f. Then F is ACG_*, i.e., [a,b] is the union of closed sets X_i on each of which F is $AC_*(X_i)$. Put $F_n(x) = F(x)$ when $x \in X_1 \cup \dots \cup X_n$ and linear or piecewise linear elsewhere. We want piecewise linearity so that $|F_n(x) - F(x)| \leq 1/n$ for all $x \in [a,b]$ and that $F_n(x)$ converges to $F(x)$ uniformly on [a,b] as $n \to \infty$.

Furthermore let $f_n(x) = F_n'(x)$ almost everywhere. It is easy to see that each f_n is Lebesgue integrable on [a,b]. Thus for each n there exists a step function ϕ_n satisfying

$$\int_a^b |f_n(x) - \phi_n(x)| dx < 2^{-n}$$

$$|f_n(x) - \phi_n(x)| < 2^{-n} \quad \text{for} \quad x \in [a,b] - E_n$$

where E_n is an open set with measure less than 2^{-n}. It is a standard argument to show that ϕ_n is control-convergent to f on [a,b]. Hence f is RD integrable on [a,b].

Therefore we have proved the following

<u>EQUIVALENCE THEOREM</u> *A function* f *is* RD *integrable on* [a,b] *if and only if it is Denjoy integrable on* [a,b].

In what follows we give a shorter proof of the controlled convergence theorem, using the definition of the RD integral.

<u>PROOF OF CONTROLLED CONVERGENCE THEOREM</u> Suppose f_n is control-convergent to f on [a,b]. Since the primitives F_n of f_n are ACG_* uniformly in n, there exists a sequence of closed sets X_i with union [a,b] and on each of which F_n is $AC_*(X_i)$ uniformly in n. Put $G_n(x) = F_n(x)$ when $x \in X_1 \cup \ldots \cup X_n$ and linear or piecewise linear (if necessary) elsewhere, and $g_n(x) = G_n'(x)$ almost everywhere. We want piecewise linearity again so that $G_n - F_n$ converges uniformly on [a,b] as $n \to \infty$. Then each g_n is Lebesgue integrable on [a,b], $g_n(x) \to f(x)$ almost everywhere in [a,b] as $n \to \infty$, and G_n is ACG_* uniformly in n. Again, there is a step function ϕ_n such that

$$\int_a^b |g_n(x) - \phi_n(x)|dx < 2^{-n}$$

$$|g_n(x) - \phi_n(x)| < 2^{-n} \quad \text{for} \quad x \in [a,b] - E_n$$

where E_n is an open set with measure less than 2^{-n}. It remains to show that ϕ_n is control-convergent to f on [a,b].

First, it is easy to see that $\phi_n(x) \to f(x)$ almost everywhere in [a,b] as $n \to \infty$. Second , let Φ_n be the primitive of ϕ_n and we see that for any $u_i, v_i \in I_i$,

$$\sum_i \left| \int_{u_i}^{v_i} \phi_n(x)dx \right| \le \sum_i \left| \int_{u_i}^{v_i} (\phi_n(x) - g_n(x))dx \right| + \sum_i \left| \int_{u_i}^{v_i} g_n(x)dx \right|$$

$$\le \int_a^b |\phi_n(x) - g_n(x)|dx + \sum_i \omega(G_n ; I_i)$$

which implies that

$$\sum_i \omega(\Phi_n; I_i) \leqq 2^{-n} + \sum_i \omega(G_n; I_i).$$

Therefore $\{\Phi_n\}$ is ACG_* uniformly in n. Finally, we write

$$||G_n - G_m|| = \sup \{ | \int_a^x (g_n - g_m)(x)dx | ; \ a \leqq x \leqq b \}$$

and we have

$$|\int_a^x (\phi_n - \phi_m)(x)dx| \leqq |\int_a^x (\phi_n - g_n)(x)| + ||G_n - G_m|| + |\int_a^x (g_m - \phi_m)(x)dx|$$

$$\leqq 2^{-n} + ||G_n - F_n|| + ||F_n - F_m|| + ||G_m - F_m|| + 2^{-m}.$$

Thus $\Phi_n(x)$ converges uniformly on [a,b]. Consequently, f is RD integrable on [a,b] and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b \phi_n(x)\,dx$$

By the construction of ϕ_n, we get

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx .$$

Hence the proof is complete.

This together with [1] and [3] provides a third proof to the controlled convergence theorem.

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