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CONTINUOUS FUNCTIONS NEED NOT HAVE σ-POROUS GRAPHS

Porosity and σ-porosity are concepts which were first introduced in [1]. The purpose of this paper is to answer a question posed by Paul Humke; namely, do continous real functions have σ-porous graphs? The following definitions will be needed:

1) A set E has porosity s at x if $\overline{\lim}\ r'/r = s$ where r' is the radius of the largest circle whose interior misses E and lies inside the circle of radius r centered at x.
2) A set E is porous if each $x \in E$ has porosity greater than zero.
3) A set E is σ-porous if E is the countable union of porous sets.

Another way to consider porosity is as follows: If B is a circle, let m*B denote the circle with the same center as B and radius m times that of B. A point $x \in E$ has porosity greater than s if and only if there are arbitrarily small circles B whose interior misses E such that x is in the interior of m*B where $s < 1/(m + 1)$.

This known equivalence can be seen by noting that if B is a ball of radius r' and x is in the interior of m*B, then if $r = mr' + r'$, B is contained in the circle of radius r centered at x. Thus for $s < \frac{1}{m+1} \frac{r'}{r}$, the porosity of E at x is greater than s. For similar reasons, the converse also holds. The porosity of a point can be no more than 1/2. Nonetheless, the following result holds for the graphs of continuous functions and, as pointed out by the referee, for all nowhere dense subsets E of R^N; namely, the set of points of porosity 1/2 of E is residual in E. To see this, consider $E' = \bigcap_{m,k} E_{mk}$ where $E_{m,k} = \{ x \in E$: there is $Z \in R^N$ with dist $(Z, x) < (1 + 1/k)$ dist $(z,E) < 1/m \}$. The ball of radius $(1 + 1/k) \cdot 2$ dist (z, E) centered at x contains the ball of radius dist (z, E) centered at Z and the second ball has no points of E in its interior. Thus, each point of E' is a point of porosity 1/2 of E and, since each E_{mk} is open and dense in E, E' is residual in E.

The example below is that of a function whose graph is non-σ-porous. The graph has Hausdorff dimension 2. It is not known whether a lower dimension number is possible or, for that matter, whether a function of bounded variation could have a non-σ-porous graph. The function $F(x) = x^{3/4}\cos(x^{-1/2})$ with $F(0) = 0$ has $(0,0)$ as a point of non-porosity of its graph and is absolutely continuous. It is not difficult to have the set of points of non-porosity be dense in the graph of a function of bounded variation. It is not clear whether this set could be uncountable.

The following lemmas will be needed in the construction of a continuous f whose graph is non-σ-porous:

Lemma 1. *Let $c_1 = 1/3$ and $c_{n+1} = c_n/(2n+1)$. Suppose that E is a subset of the plane, $x \in E$ and for $n > N$ the circle of radius c_{n+1} centered at x contains no circle of radius c_{n+3} which misses E. Then x is a point of non-porosity of E.*

Proof. Suppose E has porosity greater than $1/m$ at x. Choose n so that $m < 2n + 5$. Given $\varepsilon > 0$ with $\varepsilon < c_{n+2}$, there is a circle of radius r with $r < \varepsilon$ centered at x and a circle of radius r' which misses E and is contained in the circle of radius r and $r'/r > 1/m$. Choose p so that $c_{p+2} < r \le c_{p+1}$. Then

$$\frac{1}{m} < \frac{r'}{r} < \frac{r'}{c_{p+2}} < \frac{c_{p+3}}{c_{p+2}} = \frac{1}{2p+5} < \frac{1}{m}$$

which is a contradiction and thus the lemma is proved.

In what follows denote the graph of f on the set X by B(f;X).

Lemma 2. *Suppose* F *is defined and continuous on a perfect set* $F = F_{1,1}$ of [0,1]. *Suppose that for every pair of natural numbers* (k,n) *with* n $\geq$ k *there are perfect sets* $F_{k,n}$ *satisfying the following: Given any natural numbers* k > 1, $n_1, n_2, \ldots n_{k-1}$ with $1 \leq n_1 < n_2 < \ldots < n_{k-1}$, *let* $F_j = F_{1,n_1} \cap F_{2,n_2} \cap \ldots \cap F_{j,n_j}$. *Suppose that for all* n $> n_{k-1}$ *and for all such choices of* n_i

i) *each point of* $B(f; F_{k-1} \cap F_{k,n})$ *is a point of non-porosity of* $B(f;F_{k-1})$

ii) $F_{k-1} \cap \bigcup_{n>k} F_{k,n}$ is dense in F_{k-1}.

Then B(f;F) *is a non-σ-porous subset of the plane.*

Proof. (The proof is analogous but by no means identical to that found in [2] p. 356.) Suppose not. Then $B(f;F) = \bigcup_{n=1}^{\infty} A_n$ where each A_n is a porous subset of the plane. Let $A_0 = \emptyset$, $F_1 = F_{1,1} \cap I_1$ where $I_1 = [0,1]$ and suppose that perfect sets $F_1 \supset F_2 \supset \ldots \supset F_{k-1}$ have been chosen where $F_j = \bigcap_{i=1}^{j}(F_{i,n_i} \cap I_i)$ $j = 1,2,\ldots k-1$ and that $A_i \cap F_j = \emptyset$ for all $i < j$ and all $j \leq k - 1$.

There are two possibilities for A_{k-1}, (1) A_{k-1} is dense in F_{k-1} or (2) there is a closed interval $I_k \subset I_{k-1}$ with $A_{k-1} \cap F_{k-1} \cap I_k = \emptyset$ and $F_{k-1} \cap I_k$ perfect. In case (1), A_{k-1} is contained in the set of points of porosity of $B(f;F_{k-1})$. Then $A_{k-1} \cap B(f;F_{k,n}) = \emptyset$ for all $n > n_{k-1}$. Select any $n_k > n_{k-1}$ and any $I_k \subset I_{k-1}$ with $F_{k,n_k} \cap I_k$ perfect and let $F_k = F_{k,n_k} \cap I_k$. In case (2), with I_k determined, using ii) select $n_k > n_{k-1}$ such that $F_{k-1} \cap F_{k,n_k} \cap I_k$ is perfect and let $F_k = F_{k-1} \cap F_{k,n_k} \cap I_k$. Proceeding in this fashion yields a decreasing sequence F_k of perfect sets contained in F with $A_i \cap B(f;F_k) = \emptyset$ for all $i \leq k$ and all k. Then $\bigcap_{k=1}^{\infty} B(f;F_k)$ is not empty but contains no point of $\bigcup_{k=1}^{\infty} A_k$, a contradiction.

Example. There is a continuous real function defined on [0,1] whose graph is a non-σ-porous subset of the plane.

Construction. The function f will be defined on a perfect set F and thus can clearly be extended to [0,1] by adding to the graph monotone functions connecting the endpoints of intervals contiguous to F. Let $c_n = 1/(3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1))$. Then each $x \in [0,1]$ can be written uniquely as $\Sigma\, x_n c_n$ where the x_n are integers with $0 \leq x_n \leq 2n$. The x_n will be referred to as the places in the expansion of x. The uniqueness can be obtained by using terminating 0's whenever x has two expansions.

Certain other numbers will be used in constructions of f, $F = F_{1,1}$ and $F_{k,n}$.

Thus, let $e_n = \frac{5}{n+4} - \frac{5}{n+5}$

$P_n = 1/2 + \frac{1}{n+2}$, $N_0 = 0$, $N_1 = 18$, $N_{j+1} = N_j!$,

$K_j = N_j - N_{j-1}$.

Note that $\sum_{n=1}^{\infty} e_n = 1$, $\sum_{n=k}^{\infty} e_n = 5/(k+4)$ and

$\{p_n\} = \{5/6, 6/8, 7/10, \ldots, (n+4)/2(n+2), \ldots\}$.

Then p_n can be thought of as proportions and p_n decreases to 1/2. Furthermore, the numbers K_n are exactly divisible by $2(n+2)$ which is the p_n denominator and are divisible by all P_k denominators when $k \leq n$.

To define $f(x)$, a weight w_i will be assigned to every place in the expansions where

$$w_i = \frac{e_j}{K_j} \quad \text{when } N_{j-1} < i \leq N_j.$$

Given x and its x_i, let $a_i(x) = a_i = 1$ if x_i is odd, 0 if x_i is even. Finally, define $f(x) = \sum_{i=1}^{\infty} a_i(x) w_i$ and

$$F_{k,n} = \left\{ x \quad \begin{array}{l} \text{for } j > n \ \# \{ i \mid a_i(x)=0,\ N_{j-1} < i \leq N_j \} \leq p_k \cdot K_j \\ \text{and } \# \{ i \mid a_i(x)=0,\ N_{j-1} < i \leq N_j \} \leq p_k \cdot K_j \end{array} \right\}$$

Then $x \in F_{k,n}$ if and only if eventually (when $j > n$) x has the proportion of even x_i and the proportion of odd x_i at the places in its expansion less than or equal to p_k in each block of i where $N_{j-1} < i < N_j$. Thus each $F_{k,n}$ does not contain any points which can be written as $\sum_{i=1}^{N} x_i c_i$. It follows that if $x^m \in F_{k,n}$ and $x^m \to x$, then $x_n^m = x_n$ eventually; that is, there is M_n so that $x_n^m = x_n$ when $n > M_n$. Thus each $F_{k,n}$ is closed and since it contains no isolated points each $F_{k,n}$ is perfect. Likewise, if $x^m \to x$ in $F_{k,n}$ then $f(x^m) \to f(x)$ because of the place wise definition of f. Hence f is continuous on $F_{1,1}$ and on all $F_{k,n}$. It is clear that the $F_{k,n}$ satisfy ii) of Lemma 2. Lemma 1 will be used to show that f satisfies i) of Lemma 2 on the $F_{k,n}$ and this will prove that the graph of f is non-σ-porous. Let $k > 1$, $1 \leq n_1 < \ldots < n_{k-1} < n$, $F_{k-1} = F_{1,n_1} \cap \ldots \cap F_{k-1,n_{k-1}}$ and let $x \in F_{k-1} \cap F_{k,n}$. Suppose $j > n + 1$ and $N_j < i \leq N_{j+1}$ and consider the points $x - c_i - mc_{i+3}$ where $0 \leq m < 2c_i/c_{i+3}$. Let x' be one of these points and note that at most four places occur in x' for which the value of $a_j(x') \neq a_j(x)$. Moreover, these places occur in between

N_{j-1} and N_{j-2}. (This is because the possible values of $x_i c_i$ have numerators between 0 and 2i inclusive and thus if $x_i = 0$, $x_{i-1} = 0$, ..., $x_{i-\ell-1} = 0$ and $x_{i-\ell+1} \neq 0$ then $x' = x - c_i$ has an even term in each place x'_{i-1}, ... $x'_{i-\ell-1}$; namely, 2i, 2(i-1), ... 2(i-ℓ-1) and x' has a change of parity from x only at $x_{i-\ell+1}$. Moreover, $i - \ell + 1 > N_{j-1}$ because the proportions of even and odd places required of $F_{k,n}$ are less than 1 in each block $N_{j-1} < i < N_j$. Similarly this occurs for each $x_i' = x - c_i + mc_{i+3}$. Thus each such $x' \in F_{k-1}$. Let $x'' = x' + c_{i+1}$, then $[x', x''] \cap F_{k-1}$ is a perfect set. Note that

$$f(x) - a \leq f(x') \leq f(x) + a$$

where $a = 4e_{j-1}/K_{j-1}$. Due to the increased value of p_{k-1} to that of p_k, the function f on the interval $[x', x'']$ takes on a larger range of values on F_{k-1} than it does on $F_{k,n}$. Specifically the range of f on $[x', x''] \cap F_{k-1}$ contains points as large and as small as $f(x') \pm b$ where

$$b = \sum_{i=j+2}^{\infty} (p_{k-1} - p_k) e_i = \frac{5}{(k+2)\ (k+1)\ (j+6)} > \frac{5}{(j+6)^3}$$

Since $a = \dfrac{4.5}{(j+3)\ (j+4)K_{j-1}}$, is small in comparison to b and it is easily seen that $b - a > \dfrac{4}{(j+6)^3} > c_i$. On each interval $[x',x'']$, $f(x)$ takes on values on F_{k-1} between $f(x) - (b-a)$ and $f(x) + (b-a)$ in increments of at most $1/q$ where $q = K_{j+3}$. For if $q = K_{j+3}$ and $\frac{p}{q} \leq p_{k-1} - p_{k-1}$, then $f(x) \pm \frac{p}{q} \cdot \sum_{i=j+2}^{\infty} e_i$ is assumed on F_{k-1} on each of these intervals. Thus the ball of radius c_i centered at $(x,f(x))$ does not contain a ball of radius c_{i+3} which misses $B(f;F_{k-1})$. Since this is true at each $x \in F_{k,n}$ when $i > N_{n+1}$ it follows that i) of Lemma 2 holds and thus f has the property that its graph is non-σ-porous.

One may further note that the graph of f on F_i can be connected by means of monotone singular functions each of which is constant on a dense set of intervals in each interval continguous to F_1. On each of the intervals where the singular functions are constant, a function having small oscillation and non-σ-porous graph can be constructed. If this process is repeated, the property that its graph is non-σ-porous on every interval contained in [0,1].

REFERENCES

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